CALCULUS II

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Contents

1. Problem Set I

1. Determine the value of

$$\int_0^1 x^2 \cos(x)\ \mathrm{d}x$$

2. Find the area of the surface obtained by rotating the curve

$$y = x^3,\ 0 \le x \le 2,$$

about the x-axis.

3. Determine whether the series

$$2 - \frac{2}{\sqrt{2}} + \frac{2}{\sqrt{3}} - \frac{2}{\sqrt{4}} + \cdots$$

converges absolutely, converges conditionally, or diverges.

4. Evaluate the integral

$$\int_0^1 \frac{x^2}{(x^3+1)^2}\ \mathrm{d}x$$

5. Let $F(x) = \int_1^x \ln(t)\ \mathrm{d}t$. Find the value value $F^{''}(2)$.

6. Consider the region bounded by the graphs of $f(x) = x^2 + 1$ and $g(x) = 3x^2$. Write the integral for the volume of the solid of revolution obtained by rotating this region about the $x-$axis. Do not evaluate the integral.

7. Evaluate the integral

$$\int \frac{x+1}{x^2(x-1)} \, \mathrm{d}x$$

8. Determine whether the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}}$$

9. Use Simpson's rule with n = 6 to estimate

$$\ln(3) = \int_1^3 \frac{1}{x} \, \mathrm{d}x$$

10. Determine whether the following integral converges, and if so, evaluate it.

$$\int_0^{\infty} \cos(x) \, \mathrm{d}x$$

11. Find the length of the curve $x = \frac{1}{3}(y^2+2)^{3/2}$, from $y = 0$ to $y = 3$.

12. Determine whether the series converges or diverges. If it converges then find its sum.

$$\sum_{n=0}^{\infty} \left(\frac{5}{4^n} + \frac{(-1)^{n+1}}{3^n} \right)$$

13. Find the following approximations to

$$\int_0^{\pi/2} \cos(x) \, \mathrm{d}x$$

a. Using the **trapezoidal rule** with two intervals.

b. Using **Simpson's rule** with two intervals.

13. Find the volume of the solid of revolution formed by revolving the y-axis the region enclosed by

$$y = \cos(x^2)$$

and the $x-$axis.

1. Evaluate the following integrals.

$$a. \int \frac{\mathrm{d}x}{\sqrt{x}(1+\sqrt{x})^2} \qquad\qquad b. \int \frac{4\ \mathrm{d}x}{x^2-2x}$$

2. (a) Use the Disk method to find the volume of the solid generated by revolving the region bounded between the curve $y = 2/x, y = 0, x = 1$ and $x = 4$ about the x-axis.

b. Find the length of the curve $y = (x + \frac{5}{9})^{3/2}$ from $x = 0$ to $x = 8$.

3. Evaluate the following integrals.

$$a. \int x\ \sin(2x)\mathrm{d}x \qquad\qquad b. \int \frac{\mathrm{d}x}{(4-x^2)^{3/2}}$$

4.(a) Find the first three terms in the Taylor series of the function $f(x) = x^3 + x$ about $a = -1$.

b. Determine if the following series converges or diverges. If it converges, find its sum.

$$\sum_{n=0}^{\infty} \frac{2^{2n+2}}{5^n}$$

5(a) Use the **ratio test** to determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{n}{2^n} \frac{n!}{(n+1)!}$$

b. Use a **comparison test** to determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{n}{\sqrt{n^4+4n^2+1}}$$

6(a) Evaluate the following integral:

$$\int_1^{\infty} \frac{\mathrm{d}x}{x^2+3x+2}$$

b. A force of $F = \frac{x}{x^2+9}$ lbs is applied to move an object along the x-axis from $x = 0$ to $x = 4$ft. Determine the amount of workdone.

7(a) Evaluate the integral:

$$\int \sec^2(x)\ \tan(x)\ \mathrm{d}x$$

b. Determine the radius of convergence and interval of convergence for the power series:

$$\sum_{n=1}^{\infty} \frac{(-1)^n(x-1)^n}{\sqrt{n+1}}$$

8(a) Find the area of the polar region that lies inside one loop of the curve $r^2 = 4\ \sin(2\theta)$.

b. Find an equation for the line tangent to the curve $x = 4\ \sin(t), y = 2\ \cos(t)$ at the point $t = \pi/4$.

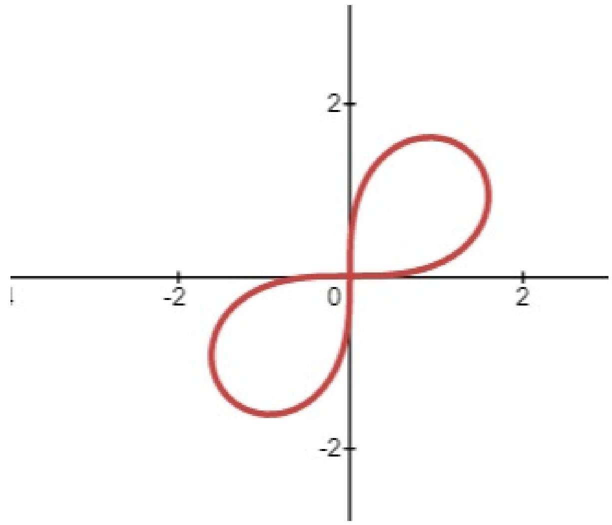

FIGURE 1. $r^2 = 4\sin(2\theta)$

3. Final Exam Solution, Fall 2021

1a. Let $u = 1 + \sqrt{x}, \mathrm{d}u = \frac{1}{2\sqrt{x}}\mathrm{d}x$.

$$\int \frac{\mathrm{d}x}{\sqrt{x}(1+\sqrt{x})^2} = \int \frac{2}{u^2}\mathrm{d}u = -\frac{2}{u} = -\frac{2}{1+\sqrt{x}}.$$

1b. Apply the partial fractions,

$$\int \frac{4\mathrm{d}x}{x^2 - 2x} = 4\int \frac{1}{2}\left[\frac{1}{x-2} - \frac{1}{x}\right]\mathrm{d}x = 2\ \ln\left|\frac{x-2}{x}\right| + C$$

2a. The volume,

$$V = \int_1^4 \pi y^2 \mathrm{d}x = \int_1^4 \pi(2/x)^2 \mathrm{d}x = 3\pi.$$

2b. The length of the curve,

$$L = \int_0^8 \sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\ \mathrm{d}x = \frac{3}{2}\int_0^8 \sqrt{1+x}\ \mathrm{d}x = 26.$$

3a.

$$\int x\ \sin(2x)\mathrm{d}x = -\frac{x}{2}\ \cos(2x) + \frac{\sin(2x)}{4} + C$$

3b. Let $x = 2\ \sin(u)$,

$$\int \frac{\mathrm{d}x}{(4-x^2)^{3/2}} = \frac{x}{4}\frac{1}{\sqrt{4-x^2}} + C$$

4a. The Taylor series expansion of $f(x)$ about $a = -1$ is given by:

$$f(x) = -2 + 4(x+1) - 3(x+1)^2.$$

4b. The given series is a geometric series with $r = \frac{4}{5} < 1$. Hence, the series converges. The sum,

$$S = \sum_{n=0}^{\infty} \frac{2^{2n+2}}{5^n} = 4\sum_{n=0}^{\infty}\left(\frac{4}{5}\right)^n = 4\ \frac{1}{1-\frac{4}{5}} = 20.$$

5a. Applying the ratio test,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{1}{2} < 1, \text{ the series converges.}$$

.

5b. Applying the limit comparison test, and comparing the given series with $\sum_{n=1}^{\infty}\frac{1}{n}$. The given series diverges.

6a.

$$\int_1^{\infty}\frac{\mathrm{d}x}{x^2+3x+2} = \lim_{a\to\infty}\int_1^{a}\frac{\mathrm{d}x}{x^2+3x+2} = \lim_{a\to\infty}\int_1^{a}\left[\frac{1}{x+1}-\frac{1}{x+2}\right]\mathrm{d}x = -\ln(2/3).$$

6b. The workdone,

$$W = \int_0^4 F\ \mathrm{d}x = \int_0^4 \frac{x}{x^2+9}\ \mathrm{d}x = \frac{1}{2}[\ln(x^2+9)]_0^4 = \ln(5/3).$$

7a.

$$\int \sec^2(x)\ \tan(x)\ \mathrm{d}x = \frac{\tan^2(x)}{2} + C$$

7b. Using Ratio test, we get the limit,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = |x-1|.$$

The series converges whenever

$$|x-1| < 1.$$

Therefore, **The radius of converges**, R=1.

Interval of convergence is where $|x-1| < R \Rightarrow |x-1| < 1 \Rightarrow 0 < x < 2$. Now we are also required to check the end points $x=0, x=2$.

Case 1: When $x=0$, the above series is

$$\sum_{n=1}^{\infty}\frac{(-1)^n(-1)^n}{\sqrt{n+1}} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}}$$

Using p-series test, the series

$$\sum_{n=1}^{\infty}\frac{1}{\sqrt{n+1}}$$

diverges.

Case 1: When $x=2$, the above series is

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$$

Using Alternating series test, the series

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$$

converges.

Therefore, **Interval of convergence** is $0 < x \le 2$.

8a. The area of the polar region,

$$A = \frac{1}{2}\int_0^{\pi/2} 4\ \sin(2\theta)\mathrm{d}\theta = \left[-\cos(2\theta)\right]_0^{\pi/2} = 2.$$

8b. Equation for tangent line:

$$(y - y_0) = \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\bigg|_{(x_0,y_0)} (x - x_0),$$

where $x_0 = 4\ \sin(\pi/4) = \frac{4}{\sqrt{2}} = 2\sqrt{2}, y_0 = 2\ \cos(\pi/4) = \sqrt{2}, \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\bigg|_{(x_0,y_0)} = \left(\frac{-2\ \sin(t)}{4\ \cos(t)}\right)_{t=\pi/4} = -\frac{1}{2}$.

Therefore, equation for tangent line,

$$\boxed{(y - \sqrt{2}) = -\frac{1}{2}\left(x - 2\sqrt{2}\right)}.$$

4. Common Exam III Solutions, Spring 2022

1(a) Determine whether the following series is convergent or divergent. Please state which test you are using.

$$\sum_{n=1}^{\infty} \frac{2+n^2}{1+2n^2}$$

Sol(1(a)) Here

$$a_n = \frac{2+n^2}{1+2n^2}, \quad \lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2+n^2}{1+2n^2} = \lim_{n\to\infty} \frac{(\frac{2}{n^2}+1)}{(\frac{1}{n^2}+2)} = \frac{1}{2} \neq 0$$

So, by **n-th term test** the series

$$\sum_{n=1}^{\infty} \frac{2+n^2}{1+2n^2}$$

diverges.

1(b) Determine whether the following series is convergent or divergent. Please state which test you are using.

$$\sum_{n=1}^{\infty} \frac{1}{n^2+\sqrt{n}}$$

Sol(1(b)) Here $a_n = \frac{1}{n^2+\sqrt{n}}$. .

Let $b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$, Find $\lim_{n\to\infty} \frac{a_n}{b_n}$,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1$$

Since,

$$\lim_{n\to\infty} \frac{a_n}{b_n} = 1 \text{ and } \sum_{n=1}^{\infty} b_n \text{ converges },$$

$$\textbf{By comparison test } \sum_{n=1}^{\infty} a_n \text{ converges }.$$

2: Find the sum of the following series:

$$(a) \sum_{n=1}^{\infty} \frac{1}{n(n+1)}, \qquad (b) \sum_{n=0}^{\infty} \frac{2^{2n+1}-6^n}{8^n}$$

Sol2(a):

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{N\to\infty} \left[\left(1-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right)\right]$$

$$= \lim_{N\to\infty} \left(1 - \frac{1}{N+1}\right) = 1$$

Sol2(b):

$$\sum_{n=0}^{\infty}\frac{2^{2n+1}-6^n}{8^n}=\sum_{n=0}^{\infty}\frac{2\times 4^n-6^n}{8^n}=2\sum_{n=0}^{\infty}\frac{4^n}{8^n}-\sum_{n=0}^{\infty}\frac{6^n}{8^n}=2\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n-\sum_{n=0}^{\infty}\left(\frac{3}{4}\right)^n=4-4=0.$$

Note: for 2(b) we are using **geometric series** sum:

$$\sum_{n=1}^{\infty} r^n=\frac{1}{(1-r)},|r|< 1.$$

3(a): Use the **integral test** to determine whether the series converges or diverges:

$$\sum_{n=3}^{\infty}\frac{1}{n(\ln(n)+1)^2}$$

Sol3(a): Let

$$f(x)=\frac{1}{x(\ln(x)+1)^2}$$

1. $f(x)$ is a continuous and non-negative function on $(3,\infty)$,
2. $f(x)$ is decreasing function on $(3,\infty)$, and

$$\int_3^{\infty}\frac{1}{x(\ln(x)+1)^2}\,\mathrm{d}x \rightarrow_{\substack{u=\ln(x)+1\\ \mathrm{d}u=\frac{1}{x}\mathrm{d}x}} \int\frac{1}{u^2}\,\mathrm{d}u=-\frac{1}{u}=-\left[\frac{1}{\ln(x)+1}\right]_3^{\infty}=\lim_{b\to\infty}-\left[\frac{1}{\ln(x)+1}\right]_3^{b}$$

$$=\lim_{b\to\infty}\left(-\frac{1}{\ln(b)+1}+\frac{1}{\ln(3)+1}\right)=\frac{1}{\ln(3)+1}$$

Since, the limit of the integral is finite. By **integral test**, the series

$$\sum_{n=3}^{\infty}\frac{1}{n(\ln(n)+1)^2}$$

converges.

3(b) Use a **comparison test** to determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty}\left(\frac{1}{n}+\left(\frac{1}{2}\right)^n\right)$$

Sol3(b) Let $b_n=\sum_{n=1}^{\infty}\frac{1}{n}$, and

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\left(1+\frac{n}{2^n}\right)=1,\text{ and }\sum_{n=1}^{\infty}b_n\text{ diverges.}$$

By comparison test $\sum_{n=1}^{\infty} a_n$ diverges

4(a) Use the **ratio test** to determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{e^{n^2}}{n!}$$

Sol4(a) Here, $a_n = \frac{e^{n^2}}{n!}$, $\qquad a_{n+1} = \frac{e^{(n+1)^2}}{(n+1)!}$

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2e \lim_{n\to\infty} e^{2n} \to \infty,$$

diverges.

4(b) Use the **root test** to determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty} \left(\frac{1}{3} + \frac{1}{n}\right)^{2n}$$

Sol(4(b)) Here,

$$a_n = \left(\frac{1}{3} + \frac{1}{n}\right)^{2n},$$

and

$$\lim_{n\to\infty} \left(a_n\right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\left(\frac{1}{3} + \frac{1}{n}\right)^{2n}\right)^{\frac{1}{n}} = \lim_{n\to\infty} \left(\frac{1}{3} + \frac{1}{n}\right)^{2} = \frac{1}{9} < 1$$

By **ratio test** the series

$$\sum_{n=1}^{\infty} \left(\frac{1}{3} + \frac{1}{n}\right)^{2n}$$

converges.

5(a): Determine whether the following series is absolutely convergent, conditionally convergent or divergent. Please state which test you are using:

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{\sqrt{n^6 + n}}$$

Sol(5(a)) Here $a_n = (-1)^n \frac{n^2}{\sqrt{n^6+n}}$, $\qquad |a_n| = \frac{n^2}{\sqrt{n^6+n}}$, let $b_n = \sum_{n=1}^{\infty} \frac{1}{n}$

$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{1}{\sqrt{1 + \frac{1}{n^5}}} = 1$$

Since the series, $\sum_{n=1}^{\infty} b_n$ diverges(by p-test). By **comparison test**, the series $\sum_{n=1}^{\infty} |a_n|$ diverges. Therefore, the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{\sqrt{n^6 + n}}$$

is not absolutely convergent.

Using the alternating series test on the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{\sqrt{n^6+n}} = \sum_{n=1}^{\infty}(-1)^n a_n,$$

we have $a_n = \frac{n^2}{\sqrt{n^6+n}}$
Observe,

1. $a_{n+1} < a_n$, for all $n \geq 1$,
2. $a_n > 0$, for all $n \geq 1$, and
3. $\lim_{n\to\infty} a_n = 0$.

Therefore, by **alternating series test** the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{n^2}{\sqrt{n^6+n}}$$

converges **conditionally.**

5(b) Determine whether the following series is absolutely convergent, conditionally convergent or divergent. Please state which test you are using:

$$\sum_{n=1}^{\infty}(-1)^n \frac{2^n}{4^n+2}$$

Sol(5(b)) Here $a_n = \sum_{n=1}^{\infty}(-1)^n \frac{2^n}{4^n+2}$, $|a_n| = \sum_{n=1}^{\infty} \frac{2^n}{4^n+2}$,

$$\text{Let } b_n = \sum_{n=1}^{\infty}\left(\frac{2}{4}\right)^n$$

Since, $|a_n| \leq b_n$, and he series, $\sum_{n=1}^{\infty} b_n$ converges(geometric series) . By **Direct comparison test**, the series $\sum_{n=1}^{\infty}|a_n|$ converges. Therefore, the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{2^n}{4^n+2}$$

is absolutely convergent.

6. Find the radius of convergence and interval of convergence for

$$\sum_{n=1}^{\infty} \frac{(x+3)^n}{n^2\ 2^n}$$

Sol(6) Using Ratio test, we get the limit,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x+3|}{2}$$

The series converges whenever

$$\frac{|x+3|}{2} < 1.$$

Therefore, **The radius of converges**, R=2.

Interval of convergence is where $|x+3| < R \Rightarrow |x+3| < 2 \Rightarrow -5 < x < -1$. Now we are also required to check the end points $x = -5, x = -1$

Case 1: When $x = -5$, the above series is

$$\sum_{n=1}^{\infty} \frac{(x+3)^n}{n^2\ 2^n} = \sum_{n=1}^{\infty} \frac{(-5+3)^n}{n^2\ 2^n} = \sum_{n=1}^{\infty} \frac{(-2)^n}{n^2\ 2^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

Using alternating series test, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

converges.

Case 2: When $x = -1$, the above series is

$$\sum_{n=1}^{\infty} \frac{(x+3)^n}{n^2\ 2^n} = \sum_{n=1}^{\infty} \frac{(-1+3)^n}{n^2\ 2^n} = \sum_{n=1}^{\infty} \frac{(2)^n}{n^2\ 2^n} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

Using **p-test**, the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges.

Therefore, **Interval of convergence** is $-5 \le x \le -1$.

7. Find the first 3 non-zero terms in the Taylor series about $a = 1$ for the function $f(x) = 3 - x + 2x^2$.

Sol(7) The Taylor series expansion is about $a = 1$ given by,

$$f(x) = f(1) + (x-1)\ f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \cdots$$

$f(1) = 4, f'(1) = 3, f''(1) = 4, f^{(n)} = 0(\text{ for all } \ n \ge 3)$

$$f(x) = 4 + 3(x-1) + \frac{4\ (x-1)^2}{2!} = 4 + 3(x-1) + 2(x-1)^2$$

8. Write down the first 3 non-zero terms in the Maclaurin series for the function $f(x) = e^{-x}\sin(2x)$.

Sol(8) The Maclaurin series expansion is given by,

$$f(x) = e^{-x}\sin(2x) = [1 - x + \frac{x^2}{2} + \cdots][2x - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \cdots]$$

Comparing the cofficients of $x^0, x^1, x^2, \cdots$, we get,

$$f(x) = 2x - 2x^2 - \frac{1}{3}x^3 + \cdots$$

9. Solve for x

$$1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^n}{n!} + \cdots = 2$$

Sol(9): Observe,

$$1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \cdots + (-1)^n \frac{x^n}{n!} + \cdots = e^{-x} = 2$$

Solving we get $x = -\ln(2)$.

5. Problem Set II

Problem 1: Find the Taylor series for e^{-x^2} centered at 0. What is the interval of convergence for this series?

Problem 2: Determine if the following series converges or diverges.

$$\sum_{n=3}^{\infty} \frac{e^{-n}}{n^2+2n}$$

Problem 3: Determine whether the following series converge or diverge

$$\sum_{n=1}^{\infty} \frac{n!}{n^n}$$

Problem 4: Determine whether the following series converge or diverge

$$\sum_{n=1}^{\infty} \frac{n+4^n}{n+6^n}$$

Problem 5: For each of the following power series, find the interval of convergence and the radius of convergence:

$a.$ $\sum_{n=1}^{\infty}(-1)^n n^2 x^n$

$b.$ $\sum_{n=1}^{\infty} \frac{2^n}{n^2}(x-3)^n$

$c.$ $\sum_{n=1}^{\infty}(-1)^n \frac{10^n}{n!}(x-10)^n$

Problem 6: Consider the function $g(x)$ defined by the power series:

$$g(x) = \sum_{n=0}^{\infty} \frac{2^n (n!)^2 x^n}{(2n)!}$$

$a.$ Find the radius of convergence of the power series.

$b.$ Use the first 3 non-zero terms of the power series to estimate

$$\int_0^1 \frac{g(x)-1}{x}\, \mathrm{d}x$$

Problem 7: determine whether the following series converge or diverge.

$$\sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$$

Problem 8: Find the Maclaurin series for $f(x) = \frac{1}{1+2x^2}$. What is the interval of convergence for this series?

6. Problem Set III

Problem 1: Find the first three terms in the Taylor Series of the function $f(x) = x^3 + x$, at $a = -1$.

Problem 2: Determine if the following series converges or diverges. If it converges, find its sum.

$$\sum_{n=0}^{\infty} \frac{2^{2n+2}}{5^n}$$

Problem 3: Use any test to check the convergence or divergence of the given series.

$$\sum_{n=1}^{\infty} \frac{n}{2^n} \frac{n!}{(n+1)!}$$

Problem 4: Use any test to check the convergence or divergence of the given series.

$$\sum_{n=1}^{\infty} \frac{(-1)^n \ (x-1)^n}{\sqrt{n+1}}$$

Problem 5: Find the power series representation for $\ln(1-x)$ and its radius of convergence.

Problem 6: Find the first three nonzero terms in the Maclaurin series for $f(x) = e^x \sin(x)$.

Problem 7: Use any test to check the convergence or divergence of the given series.

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$$

7. Problem Set IV

Problem 1: Determine whether the series is absolutely convergent, conditionally convergent, or divergent.

$(a)\ \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln\ n}$

$(b)\ \sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^{n^2}$

$(c)\ \sum_{n=1}^{\infty} \frac{10^n}{(n+1)4^{2n+1}}$

Problem 2: Find the **radius of convergence and interval of convergence** of the series.

$(a)\ \sum_{n=0}^{\infty} \frac{(x-2)^n}{n^2+1}$

$(b)\ \sum_{n=1}^{\infty} \frac{(5x-4)^n}{n^3}$

$(c)\ \sum_{n=1}^{\infty} \frac{n}{b^n}(x-a)^n,\ b>0$

3(a) Use the **ratio test** to determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty} \frac{(n+2)!}{n!\ 9^n}$$

3(b) Use the **root test** to determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}+\frac{1}{3n}\right)^n$$

4(a): Determine whether the following series is absolutely convergent, conditionally convergent or divergent. Please state which test you are using:

$$\sum_{n=1}^{\infty} (-1)^n \frac{n}{\sqrt{4n^3+n}}$$

4(b) Determine whether the following series is absolutely convergent, conditionally convergent or divergent. Please state which test you are using:

$$\sum_{n=1}^{\infty} (-1)^n \frac{e^n}{e^{2n}+1}$$

5(a) Determine whether the following series is convergent or divergent. Please state which test you are using.

$$\sum_{n=1}^{\infty} \left(\frac{2+n^2}{1+2n^2}\right)^n$$

5(b) Determine whether the following series is convergent or divergent. Please state which test you are using.

$$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^{1/n}$$

6(a) Write down the first 3 non-zero terms in the Maclaurin series for the function $f(x) = x + \cos(2x)$.

6(b) Find the first 3 non-zero terms in the Taylor series about $a = 1$ for the function $f(x) = 2 - x^2$.

7. Find the radius of convergence and interval of convergence for

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{n\ 3^n}$$

8(a): Solve for x

$$1 + x + x^2 + x^3 + \cdots = 2$$

8(b): Find the Taylor polynomial or order 2 generated by $f(x) = \ln(x)$ about $a = 1$.

8. Problem Set V

Problem 1: Find the value of p for which the series is convergent.

$$\sum_{n=2}^{\infty} \frac{1}{n(\text{ln n})^{\text{p}}}$$

Problem 2: Find the value of p for which the series is convergent.

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n[\ln(\text{ln n})]^p}$$

Problem 3: Find the value of p for which the series is convergent.

$$\sum_{n=1}^{\infty} \frac{\ln n}{n^p}$$

Problem 4: Find the interval of convergence for $f, f',$ and f''.

$$f(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^2}$$

Problem 5: Find the value of p for which the series is convergent.

$$\sum_{n=1}^{\infty} n(1+n^2)^p$$

Problem 6: Find the value of the integral:

$$\int_0^{1/2} \frac{\mathrm{d}x}{x^2 - x + 1}$$

Problem 7: Find the sum of the following series.

$$(i) \sum_{n=2}^{\infty} n(n-1)x^n, |x| < 1, \qquad (ii) \sum_{n=2}^{\infty} \frac{n^2 - n}{2^n} \qquad (iii) \sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

9. Problem Set VI

Problem 1: Apply the root test to the following series.

$$\sum_{n=1}^{\infty} \left(\frac{n}{2n+3} \right)^n$$

Problem 2: Apply the root test to the following series.

$$\sum_{n=1}^{\infty} \frac{2^n}{n^{2n}}$$

Problem 3: Apply the ratio test to the following alternating series.

$$\sum_{n=1}^{\infty} (-1)^n \frac{n!}{1000^n}$$

Problem 4: Evaluate the following series using any test.

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{n^2}$$

Problem 5: Evaluate the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n^3 - n^2}}$$

Problem 6: Evaluate the following series converges or diverges.

$$\sum_{n=1}^{\infty} (0.8)^{-n}\ n^{-0.8}$$

Problem 7: Evaluate the following series converges or diverges.

$$\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$$

Problem 8: State whether the following series converges or diverges.

$$\sum_{n=2}^{\infty} \frac{1}{n\sqrt{\ln(n)}}$$

10. Problem Set VII

1. Integrate

$$\int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x$$

2. Integrate

$$\int \frac{x^3}{\sqrt{1-x^2}}\,\mathrm{d}x$$

3. Integrate

$$\int \frac{1}{(4x^2+9)^2}\,\mathrm{d}x$$

4. Integrate

$$\int x\,\tan^{-1}(x)\,\mathrm{d}x$$

11. Problem Set VIII

Q1 Find the following integrals:

$(a)\int \frac{3x^2-6}{(x-1)^3}\ \mathrm{d}x$

$(b)\int \frac{\sqrt{x^2-9}}{x}\ \mathrm{d}x$

$(c)\int \frac{x^4-2x^3+4x^2+4}{x^3+x}\ \mathrm{d}x$

$(d)\int \frac{x^3}{(4x^2+1)^{\frac{3}{2}}}\ \mathrm{d}x$

$(e)\int \frac{2\ \sin^{-1}(x)}{x^3}\ \mathrm{d}x$

12. Solutions to Problem Set VIII

1a.

$$\begin{aligned}\int \frac{3x^2-6}{(x-1)^3}\ \mathrm{d}x \Rightarrow_{\substack{u=x-1\\ \mathrm{d}u=\mathrm{d}x}} 3\int \frac{(u+1)^2-2}{u^3}\ \mathrm{d}u &= 3\int \frac{u^2+2u-1}{u^3}\ \mathrm{d}u\\ &= 3\Bigg[\int \frac{1}{u}\ \mathrm{d}u + \int \frac{2}{u^2}\ \mathrm{d}u - \int \frac{1}{u^3}\ \mathrm{d}u\Bigg]\\ &= 3\Bigg[\ln|u| - \frac{2}{u} + \frac{1}{2u^2}\Bigg]\\ &= 3\Bigg[\ln|x-1| - \frac{2}{x-1} + \frac{1}{2(x-1)^2}\Bigg] + C\end{aligned}$$

1b.

$$\int \frac{\sqrt{x^2-9}}{x}\ \mathrm{d}x \Rightarrow_{\substack{u=\sqrt{x^2-9}\\ \mathrm{d}u=\frac{x}{\sqrt{x^2-9}}\mathrm{d}x}} \int \frac{\sqrt{x^2-9}}{x}\frac{\sqrt{x^2-9}}{x}\ \mathrm{d}x = \int \frac{x^2-9}{x^2}\ \mathrm{d}x = \int \frac{(u^2+9)-9}{u^2+9}\mathrm{d}u$$

$$= \int 1\ \mathrm{d}u - 9\int \frac{1}{u^2+9}\ \mathrm{d}u = u - 9\bigg(\frac{1}{3}\tan^{-1}\Big(\frac{u}{3}\Big)\bigg) = \sqrt{x^2-9}\ -\ 3\ \tan^{-1}\bigg(\frac{\sqrt{x^2-9}}{3}\bigg) + C$$

1c.

$$\int \frac{x^4-2x^3+4x^2+4}{x^3+x}\ \mathrm{d}x = \int \frac{x(x^3-2x^2)+4(x^2+1)}{x(x^2+1)}\ \mathrm{d}x = \int \frac{x^3-2x^2}{x^2+1}\ \mathrm{d}x + \int \frac{4}{x}\mathrm{d}x$$

Apply the long division to find the integral

$$\int \frac{x^3-2x^2}{x^2+1}\ \mathrm{d}x = \int \frac{x^2(x-2)}{x^2+1}\ \mathrm{d}x = \int (x-2)\bigg(1-\frac{1}{x^2+1}\bigg)\ \mathrm{d}x = \int (x-2)\ \mathrm{d}x - \int \frac{(x-2)}{x^2+1}\ \mathrm{d}x$$

$$\begin{aligned}&= \int (x-2)\ \mathrm{d}x - \int \frac{x}{x^2+1}\ \mathrm{d}x + \int \frac{2}{x^2+1}\ \mathrm{d}x\\ &= \frac{x^2}{2} - 2x - \frac{1}{2}\ln|x^2+1| + 2\tan^{-1}(x)\end{aligned}$$

$$\int \frac{x^4-2x^3+4x^2+4}{x^3+x}\ \mathrm{d}x = \frac{x^2}{2} - 2x - \frac{1}{2}\ln|x^2+1| + 2\tan^{-1}(x) + 4\ln|x| + C$$

1d.

$$\int \frac{x^3}{(4x^2+1)^{\frac{3}{2}}}\ \mathrm{d}x \Rightarrow_{\substack{u=4x^2+1\\ \mathrm{d}u=8x\mathrm{d}x}} \frac{1}{32}\int \frac{u-1}{u^{3/2}}\mathrm{d}u = \frac{1}{32}(2\sqrt{u} + \frac{2}{\sqrt{u}}) = \frac{\sqrt{4x^2+1}}{16} + \frac{1}{16\sqrt{4x^2+1}} +\ C$$

1e.

$$\int \frac{2\ \sin^{-1}(x)}{x^3}\ \mathrm{d}x = 2\left(\sin^{-1}(x)\int \frac{1}{x^3}\ \mathrm{d}x - \int \frac{\mathrm{d}}{\mathrm{d}x}(\sin^{-1}(x))\int \frac{1}{x^3}\ \mathrm{d}x\right)$$

$$= \frac{-\sin^{-1}(x)}{x^2} + \int \frac{1}{x^2\sqrt{x^2-1}}\ \mathrm{d}x = -\left[\frac{\sin^{-1}(x)}{x^2} + \tan(\cos^{-1}(x))\right] + C$$

$$\int \frac{1}{x^2\sqrt{1-x^2}}\ \mathrm{d}x \Rightarrow_{\substack{x=\cos(u)\\ \mathrm{d}x=-\sin(u)\ \mathrm{d}u}} = \int \frac{-\sin(u)\ \mathrm{d}u}{\cos^2(u)\sqrt{1-\cos^2(u))}} = \int \frac{-\sin(u)\ \mathrm{d}u}{\cos^2(u)\ \sin(u)}$$

$$= -\int \frac{1}{\cos^2(u)} = -\int \sec^2(u) = -\tan(u) = -\tan(\cos^{-1}(x))$$

13. Problem Set IX

1. Determine whether the following sequences $\{a_n\}$ are convergent or divergent. Find the limit of any convergent sequences.

a. $a_n = \left(\dfrac{n^3 + 5n^4}{2n^4 + 2n - 1}\right)^{\frac{1}{3}}$ **b.** $a_n = n \sin\left(\dfrac{1}{n}\right)$

2. Evaluate

$$\int \frac{1}{(1+x^2)^{\frac{5}{2}}}\ \mathrm{d}x$$

3. Check the convergence or divergence of the following sequences:

a. $a_n = \log(2n^2 + 1) - 2\ \log(n)$ **b.** $a_n = ne^{-n}$

4. Check the convergence or divergence of the following integral:

$$\int_1^{\infty} \frac{1}{(x^2 + 3x + 2)}\ \mathrm{d}x$$

5. Evaluate

$$\int \ln(x^2 + 1)\ \mathrm{d}x$$

6. Check the convergence or divergence of the following sequences:

a. $a_n = \log(n^2 + 1) - 3\ \log(n)$ **b.** $a_n = \sqrt{n}\left(1 - \cos\left(\dfrac{1}{n}\right)\right)$

7. Consider the integral

$$\int_2^4 \frac{1}{2x - 3}\ \mathrm{d}x.$$

Estimate the integral using the **trapezoidal** rule with $n = 4$ steps.

8. Evaluate the following integrals if they are convergent or show they are divergent:

a. $\displaystyle\int_0^{\pi} \tan^2(x)\ \sec^2(x)\ \mathrm{d}x$ **b.** $\displaystyle\int_1^{\infty} \frac{x}{x^2 + 1}\ \mathrm{d}x$

1. Determine if the following sequences converges or diverges. If converges find the limit.

a.

$$a_n = \frac{\sqrt{n+4n^2}}{2+n} = \left[\frac{\sqrt{\frac{1}{n}+4}}{\frac{2}{n}+1}\right]_{n\to\infty} = \sqrt{4} = 2$$

b.

$$a_n = (2)^{\frac{1}{n}} \Rightarrow \ln(a_n) = \ln(2)^{\frac{1}{n}} \Rightarrow \ln(a_n) = \frac{1}{n}\ln(2)$$

Applying the limits both sides, and taking exponential we get:

$$\lim_{n\to\infty} a_n = 1.$$

2. Evaluate the integrals:

a.

$$\int \cos^3(\theta)\ \mathrm{d}\theta = \int \cos(\theta)\cos^2(\theta) = \int \cos(\theta)(1-\sin^2(\theta)) = \int \cos(\theta) - \int \cos(\theta)\sin^2(\theta)$$

$$= \sin(\theta) - \frac{\sin^3(\theta)}{3} + C$$

b.

$$\int \frac{\ln x^2}{x^2}\ \mathrm{d}x = 2\int \frac{\ln x}{x^2}\ \mathrm{d}x = -2\left[\frac{\ln x}{x} + \frac{1}{x}\right] + C$$

Integrate by parts:

$$\int \frac{\ln x}{x^2}\ \mathrm{d}x = \ln x \int \frac{1}{x^2}\mathrm{d}x - \int \frac{\mathrm{d}}{\mathrm{d}x}\ln x \int \frac{1}{x^2}\mathrm{d}x = -\frac{\ln x}{x} + \int \frac{1}{x^2}\mathrm{d}x = -\left[\frac{\ln x}{x} + \frac{1}{x}\right] + C$$

3. Evaluate the Integrals:

a.

$$\int \sqrt{4-x^2}\ \mathrm{d}x \Rightarrow_{\substack{x=2\ \sin u \\ \mathrm{d}x=2\ \cos u\ \mathrm{d}u}} 4\int \cos^2(u)\ \mathrm{d}u = 2\int (1+\cos(2u))\ \mathrm{d}u = 2\left[u + \frac{\sin(2u)}{2}\right]$$

$$= 2\left[\sin^{-1}\left(\frac{x}{2}\right) + \frac{1}{2}\left(\sin\left(2\sin^{-1}\frac{x}{2}\right)\right)\right] + C$$

b.

$$\int (1+\tan(x))\cos(x)\ \mathrm{d}x = \int (\cos(x)+\sin(x))\mathrm{d}x = -\sin(x) + \cos(x) + C$$

4a.

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-16}} \Rightarrow_{\substack{x=4\sec(u) \\ \mathrm{d}x=4\ \sec(u)\tan(u)\mathrm{d}u}} \int \frac{\sec(u)\tan(u)\mathrm{d}u}{\sqrt{\sec^2(u)-1}} = \int \sec(u)\ \mathrm{d}u = (\ln|\tan(u)+\sec(u)|)$$

$$= \ln\left|\tan\left(\sec^{-1}\left(\frac{x}{4}\right)\right) + \sec\left(\sec^{-1}\left(\frac{x}{4}\right)\right)\right| + C$$

4b.

$$\int \frac{3x-5}{x^2-3x+2}\ \mathrm{d}x = \int \left(\frac{1}{x-2} + \frac{2}{x-1}\right)\ \mathrm{d}x = \ln|x-2| + 2\ \ln|x-1| + C$$

5(a)

$$\int \sec^3(\theta)\ \tan(\theta)\ \mathrm{d}\theta \Rightarrow_{\substack{u=\sec(\theta) \\ \mathrm{d}u=\sec(\theta)\tan(\theta)\ \mathrm{d}\theta}} \int u^2\ \mathrm{d}u = \frac{u^3}{3} = \frac{\sec^3(\theta)}{3} + C$$

5(b)

$$\int \frac{x}{x^4+1}\,\mathrm{d}x = \int \frac{x}{(x^2)^2+1}\,\mathrm{d}x \Rightarrow_{\mathrm{d}u=2x\mathrm{d}x}^{u=x^2} = \frac{1}{2}\int \frac{1}{u^2+1}\,\mathrm{d}u = \frac{1}{2}\tan^{-1}(u) = \frac{1}{2}\tan^{-1}(x^2)+C$$

6.(a) Apply the Direct comparison test to determine if the following integral converges.

$$\int_0^1 \frac{1}{x^2+\sqrt{x}}\,\mathrm{d}x \le \int_0^1 \frac{1}{\sqrt{x}}\,\mathrm{d}x = \lim_{a\to 0} 2\left[\sqrt{x}\right]_a^1 = 2,\ \text{convergent}$$

6(b). Apply the Limit Comparison test to determine if the following integral converges.

$$\int_1^\infty \frac{e^x}{x\sqrt{e^{2x}+4}}$$

$$\lim_{x\to\infty} \frac{\frac{e^x}{x\sqrt{e^{2x}+4}}}{1/x} = 1,\ \text{and}\quad \int_1^\infty \frac{1}{x}\mathrm{d}x = \textbf{Diverges},\ \textbf{therefore above integral diverges.}$$

7. Find the step size,

$$\boxed{h = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}}$$

$$\boxed{x_0 = 0,\ x_1 = \frac{1}{2},\ x_2 = 1,\ x_3 = \frac{3}{2},\ x_4 = 2}$$

$$\boxed{f(0) = -1,\ f(1/2) = -7/8,\ f(1) = 0,\ f(3/2) = 19/8,\ f(2) = 7}$$

Using Trapezoidal rule:

$$\int_0^2 (x^3-1)\,\mathrm{d}x = \frac{h}{2}\left[f(0) + 2\Big(f(1/2)+f(1)+f(3/2)\Big) + f(2)\right] = \frac{9}{4}.$$

Error bound,

$$E = \frac{\max|f'''(x)|(b-a)^3}{12n^2} = \frac{6.(2-0)^3}{12.4^2} = \frac{1}{4}.$$

8. Evaluate the improper integrals:

a.

$$\int_0^\infty \frac{e^{-x}}{1+e^{-x}}\,\mathrm{d}x = \lim_{a\to\infty}\int_0^a \frac{e^{-x}}{1+e^{-x}}\,\mathrm{d}x \Rightarrow_{\mathrm{d}u=-e^{-x}\mathrm{d}x}^{u=1+e^{-x}} = \lim_{a\to\infty}\left[-\ln|1+e^{-x}|\right]_0^a = \ln(2).$$

b.

$$\int_1^2 \frac{1}{\sqrt{x-1}}\,\mathrm{d}x = \lim_{a\to 1}\int_a^2 \frac{1}{\sqrt{x-1}}\,\mathrm{d}x = \lim_{a\to 1}\left[2\sqrt{x-1}\right]_a^2 = 2.$$

9.

$$\int_0^{\pi/4} \frac{\sec^4(x)}{\sqrt{\tan x}}\,\mathrm{d}x \Rightarrow_{\mathrm{d}u=\sec^2(x)\mathrm{d}x}^{u=\tan x} \int_0^1 \frac{1+u^2}{\sqrt{u}}\mathrm{d}u = \left[2\sqrt{u}+\frac{2}{5}u^{5/3}\right]_0^1 = \frac{12}{5}.$$

15. Common Exam III Solutions, Fall 2021

1: Find the sum of the following series:

$$(a)\sum_{n=3}^{\infty}\frac{1}{(n-1)(n-2)}, \qquad (b)\ \sum_{n=1}^{\infty}\frac{2^{2n}+4^n}{6^n}$$

Sol1(a):

$$\sum_{n=3}^{\infty}\frac{1}{(n-1)(n-2)}=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\cdots=\lim_{n\to\infty}\left[\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\cdots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$$

$$=\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)=1$$

Sol1(b):

$$\sum_{n=1}^{\infty}\frac{2^{2n}+4^n}{6^n}=\sum_{n=1}^{\infty}\frac{4^n+4^n}{6^n}=2\sum_{n=1}^{\infty}\frac{4^n}{6^n}=2\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n=\frac{2}{(1-\frac{2}{3})}=6$$

Note: for 1(b) we are using **geometric series** sum:

$$\sum_{n=1}^{\infty}r^n=\frac{1}{(1-r)}, |r|< 1.$$

2(a): Use the **integral test** to determine whether the series converges or diverges:

$$\sum_{n=2}^{\infty}\frac{1}{n(\ln(n))^2}$$

Sol2(a): Let

$$f(x)=\frac{1}{x(\ln(x))^2}$$

1. $f(x)$ is a continuous and non-negative function on $(2,\infty)$,
2. $f(x)$ is non-decreasing function on $(2,\infty)$, and

$$\int_2^{\infty}\frac{1}{x(\ln(x))^2}\ \mathrm{d}x \xrightarrow[\mathrm{d}u=\frac{1}{x}\mathrm{d}x]{u=\ln(x)} \int\frac{1}{u^2}\ \mathrm{d}u=-\frac{1}{u}=-\left[\frac{1}{\ln(x)}\right]_2^{\infty}=\lim_{b\to\infty}-\left[\frac{1}{\ln(x)}\right]_2^{b}$$

$$=\lim_{b\to\infty}\left(-\frac{1}{\ln(b)}+\frac{1}{\ln(2)}\right)=\frac{1}{\ln(2)}$$

Since, the limit of the integral is finite. By **integral test**, the series

$$\sum_{n=2}^{\infty}\frac{1}{n(\ln(n))^2}$$

converges.

2(b) Use a **comparison test** to determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{4}\right)^n$$

Sol2(b) Let $b_n = \sum_{n=1}^{\infty}\left(\frac{1}{4}\right)^n$, Find $\lim_{n\to\infty}\frac{a_n}{b_n}$, where $a_n = \frac{1}{n}\left(\frac{1}{4}\right)^n$

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{1}{n} = 0$$

Since,

$$\lim_{n\to\infty}\frac{a_n}{b_n} = 0 \text{ and } \sum_{n=1}^{\infty} b_n \text{ converges } \left(\text{ geometric series with } r = \frac{1}{4}\right),$$

$$\textbf{By comparison test } \sum_{n=1}^{\infty} a_n \text{ converges .}$$

3(a) Use the **ratio test** to determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty}\frac{(n+2)!}{n!\ 9^n}$$

Sol3(a) Here, $a_n = \frac{(n+2)!}{n!\ 9^n}$, $\quad a_{n+1} = \frac{(n+3)!}{(n+1)!\ 9^{n+1}}$

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(n+3)!\ (n)!\ 9^n}{(n+2)!(n+1)!\ 9^{n+1}} = \frac{1}{9}\lim_{n\to\infty}\frac{(n+3)}{(n+1)} = \frac{1}{9}\lim_{n\to\infty}\frac{(1+\frac{3}{n})}{(1+\frac{1}{n})} = \frac{1}{9} < 1$$

By **ratio test** the series

$$\sum_{n=1}^{\infty}\frac{(n+2)!}{n!\ 9^n}$$

converges.

3(b) Use the **root test** to determine whether the series converges or diverges:

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}+\frac{1}{3n}\right)^n$$

Sol(3(b)) Here,

$$a_n = \left(\frac{1}{2}+\frac{1}{3n}\right)^n,$$

and

$$\lim_{n\to\infty}\left(a_n\right)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\left(\frac{1}{2}+\frac{1}{3n}\right)^n\right)^{\frac{1}{n}} = \lim_{n\to\infty}\left(\frac{1}{2}+\frac{1}{3n}\right) = \frac{1}{2} < 1$$

By **ratio test** the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}+\frac{1}{3n}\right)^n$$

converges.

4(a): Determine whether the following series is absolutely convergent, conditionally convergent or divergent. Please state which test you are using:

$$\sum_{n=1}^{\infty}(-1)^n \frac{n}{\sqrt{4n^3+n}}$$

Sol(4(a)) Here $a_n = (-1)^n \frac{n}{\sqrt{4n^3+n}}$, $\quad |a_n|= \frac{n}{\sqrt{4n^3+n}}$, let $b_n = \sum_{n=1}^{\infty} \frac{1}{n^{\frac{1}{2}}}$

$$\lim_{n\to\infty} \frac{|a_n|}{b_n} = \lim_{n\to\infty} \frac{n\ n^{\frac{1}{2}}}{\sqrt{4n^3+n}} = \lim_{n\to\infty} \frac{n\ n^{\frac{1}{2}}}{n^{\frac{3}{2}}\sqrt{4+\frac{1}{n^2}}} = \lim_{n\to\infty} \frac{1}{\sqrt{4+\frac{1}{n^2}}} = \frac{1}{2} > 0$$

Since the series, $\sum_{n=1}^{\infty} b_n$ diverges(by p-test). By **comparison test**, the series $\sum_{n=1}^{\infty}|a_n|$ diverges. Therefore, the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{n}{\sqrt{4n^3+n}}$$

is not absolutely convergent.

Using the alternating series test on the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{n}{\sqrt{4n^3+n}} = \sum_{n=1}^{\infty}(-1)^n a_n,$$

we have $a_n = \frac{n}{\sqrt{4n^3+n}}$
Observe,

1. $a_{n+1} < a_n$, for all $n \geq 1$,
2. $a_n > 0$, for all $n \geq 1$, and
3. $\lim_{n\to\infty} a_n = 0$.

Therefore, by **alternating series test** the series

$$\sum_{n=1}^{\infty}(-1)^n \frac{n}{\sqrt{4n^3+n}}$$

converges **conditionally.**

4(b) Determine whether the following series is absolutely convergent, conditionally convergent or divergent. Please state which test you are using:

$$\sum_{n=1}^{\infty}(-1)^n \frac{e^n}{e^{2n}+1}$$

Sol(4(b)) Here $a_n = \sum_{n=1}^{\infty}(-1)^n \frac{e^n}{e^{2n}+1}$, $\quad |a_n|= \sum_{n=1}^{\infty} \frac{e^n}{e^{2n}+1}$,

$$\text{Let } b_n = \sum_{n=1}^{\infty} e^n$$

$$\lim_{n\to\infty}\frac{|a_n|}{b_n}=\lim_{n\to\infty}\frac{e^n}{e^n(1+e^{2n})}=\lim_{n\to\infty}\frac{1}{1+e^{2n}}=1>0$$

The series, $\sum_{n=1}^{\infty} b_n$ diverges using by limit test. Since

$$\lim_{n\to\infty} b_n=\lim_{n\to\infty} e^n\neq 0$$

By **comparison test**, the series $\sum_{n=1}^{\infty}|a_n|$ diverges. Therefore, the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{e^n}{e^{2n}+1}$$

is not absolutely convergent.

Using alternating series test on the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{e^n}{e^{2n}+1}=\sum_{n=1}^{\infty}(-1)^n\ a_n$$

we have $a_n=\frac{e^n}{e^{2n}+1}$
Observe,

1. $a_{n+1}<a_n$, for all $n\geq 1$,
2. $a_n>0$, for all $n\geq 1$, and
3. $\lim_{n\to\infty} a_n=0$.

Therefore, by **alternating series test** the series

$$\sum_{n=1}^{\infty}(-1)^n\frac{e^n}{e^{2n}+1}$$

converges **conditionally.**

5(a) Determine whether the following series is convergent or divergent. Please state which test you are using.

$$\sum_{n=1}^{\infty}\left(\frac{2+n^2}{1+2n^2}\right)^n$$

Sol(5(a)) Here

$$a_n=\left(\frac{2+n^2}{1+2n^2}\right)^n,\quad \lim_{n\to\infty}\bigg(a_n\bigg)^{1/n}=\lim_{n\to\infty}\frac{2+n^2}{1+2n^2}=\lim_{n\to\infty}\frac{(\frac{2}{n^2}+1)}{(\frac{1}{n^2}+2)}=\frac{1}{2}<1$$

So, by **root test** the series

$$\sum_{n=1}^{\infty}\left(\frac{2+n^2}{1+2n^2}\right)^n$$

converges.

5(b) Determine whether the following series is convergent or divergent. Please state which test you are using.

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{1/n}$$

Sol(5(b)) Here $a_n = \left(\frac{1}{2}\right)^{1/n}$. Find the $\lim_{n\to\infty} a_n$.

Let $y = a_n = \left(\frac{1}{2}\right)^{1/n} \Rightarrow \ln(y) = \ln\left(\frac{1}{2}\right)^{1/n} \Rightarrow \ln(y) = \frac{1}{n}\ln\left(\frac{1}{2}\right)$, take limits both sides

$$\lim_{n\to\infty}\ln(y) = \lim_{n\to\infty}\frac{1}{n}\ln\left(\frac{1}{2}\right) \Rightarrow \lim_{n\to\infty}\ln(y) = 0$$

take exponential both sides

$$\lim_{n\to\infty} e^{\ln(y)} = e^0 \Rightarrow \lim_{n\to\infty} y = \lim_{n\to\infty} a_n = 1 \neq 0$$

By **limit test**, the series

$$\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^{1/n}$$

diverges.

6(a) Write down the first 3 non-zero terms in the Maclaurin series for the function $f(x) = x + \cos(2x)$.

Sol(6(a)) The Maclaurin series expansion is given by,

$$f(x) = f(0) + x\ f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots$$

$f(0) = 1,\ f'(0) = 1, f''(0) = -4, \cdots\cdots\cdots$

$$f(x) = 1 + x + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(-4) + \cdots$$

Therefore, the first 3 non-terms are

$$a_0 = 1, a_1 = 1, a_2 = \frac{1}{2}.$$

6(b) Find the first 3 non-zero terms in the Taylor series about $a = 1$ for the function $f(x) = 2 - x^2$.

Sol(6(b)) The Taylor series expansion is about $a = 1$ given by,

$$f(x) = f(1) + (x-1)\ f'(1) + \frac{(x-1)^2}{2!}f''(1) + \frac{(x-1)^3}{3!}f'''(1) + \cdots$$

$f(1) = 1, f'(1) = -2, f''(1) = -2, f^{(n)} = 0(\forall\ n \geq 3)$

$$f(x) = 1 - 2(x-1) - \frac{2\ (x-1)^2}{2!} = 1 - 2(x-1) - (x-1)^2$$

The first, second and third non-zero terms are the coefficients of $(x-1)^0, (x-1)^1, (x-1)^2$ respectively. Therefore, the first 3 non-zero terms are

$$a_0 = 1, a_1 = -2, a_3 = -1.$$

7. Find the radius of convergence and interval of convergence for

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{n\ 3^n}$$

Sol(7) The radius of convergence, R, is given by

$$\frac{1}{R} = \lim_{n\to\infty} (a_n)^{1/n}$$

For the given problem, $a_n = \frac{1}{n\ 3^n} \to (a_n)^{1/n} = \frac{1}{3n^{1/n}}$

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{1}{3n^{1/n}} = \frac{1}{3} \lim_{n\to\infty} \frac{1}{n^{1/n}}$$

$$\text{let } y = n^{1/n} \Rightarrow \ln(y) = \ln(n^{1/n}) \Rightarrow \ln(y) = \frac{1}{n}\ln(n)$$

Now take the limits both sides,

$$\lim_{n\to\infty} \ln(y) = \lim_{n\to\infty} \frac{\ln(n)}{n} \left[\frac{\infty}{\infty}(\text{ form, use L'Hospital's Rule })\right] \Rightarrow \lim_{n\to\infty} \ln(y) = \lim_{n\to\infty} \frac{1}{n} = 0$$

Taking exponential both sides gives

$$\lim_{n\to\infty} y = \lim_{n\to\infty} n^{1/n} = 1$$

Therefore, the limit

$$\lim_{n\to\infty} (a_n)^{1/n} = \frac{1}{3} = \frac{1}{R} \Rightarrow R = 3.$$

Interval of convergence is where $|x+2| < R \Rightarrow |x+2| < 3 \Rightarrow -5 < x < 1$. Now we are also required to check the end points $x = -5, x = 1$

Case 1: When $x = -5$, the above series is

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{n\ 3^n} = \sum_{n=1}^{\infty} \frac{(-5+2)^n}{n\ 3^n} = \sum_{n=1}^{\infty} \frac{(-3)^n}{n\ 3^n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

Using alternating series test, the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

converges.

Case 2: When $x = 1$, the above series is

$$\sum_{n=1}^{\infty} \frac{(x+2)^n}{n\ 3^n} = \sum_{n=1}^{\infty} \frac{(1+2)^n}{n\ 3^n} = \sum_{n=1}^{\infty} \frac{(3)^n}{n\ 3^n} = \sum_{n=1}^{\infty} \frac{1}{n}$$

Using **p-test**, the series

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

Therefore, **Interval of convergence** is $-5 \leq x < 1$.

8(a): Solve for x

$$1 + x + x^2 + x^3 + \cdots = 2$$

Sol(8(a)) Observe,

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n = \frac{1}{1-x} = 2$$

Solving we get $x = \frac{1}{2}$.

8(b): Find the Taylor polynomial or order 2 generated by $f(x) = \ln(x)$ about $a = 1$.

Sol(8(b)) The Taylor series expansion is about $a = 1$ given by,

$$f(x) = f(1) + (x-1)\ f'(1) + \frac{(x-1)^2}{2!} f''(1) + \frac{(x-1)^3}{3!} f'''(1) + \cdots$$

$f(1) = 0, f'(1) = 1, f''(1) = -1, f'''(1) = -2, \cdots\cdots\cdots$

$$f(x) = (x-1) - \frac{(x-1)^2}{2!} - \frac{2(x-1)^3}{3!} + \cdots\cdots\cdots$$

Therefore, the Taylor polynomial of order 2,

$$P_2(x) = (x-1) - \frac{(x-1)^2}{2!}\ .$$

16. Common Exam I Solutions, Spring 2022

Q1: Find the length of the curve $y = \frac{1}{6}(2+4x^2)^{3/2}$ over $0 \leq x \leq 3$.

Sol(1): The length of the curve,

$$L = \int_0^3 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, \mathrm{d}x$$

$$\frac{dy}{dx} = 2x\sqrt{2+4x^2} \Rightarrow \left(\frac{dy}{dx}\right)^2 = 4x^2(2+4x^2) \Rightarrow \left(1+\left(\frac{dy}{dx}\right)^2\right) = (1+4x^2)^2$$

$$L = \int_0^3 \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx = \int_0^3 (1+4x^2)\mathrm{d}x = \left[x+\frac{4}{3}x^3\right]_0^3 = 39.$$

Q2: Find the area of the surface formed by rotating the curve $y = \sqrt{6x - x^2}$ for $1 \leq x \leq 3$ about x-axis.

Sol(2): The area of the surface,

$$S = 2\pi \int_1^3 y(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, \mathrm{d}x$$

$$\left(\frac{dy}{dx}\right) = \frac{1}{2\sqrt{6x-x^2}}(6-2x) \Rightarrow \left(\frac{dy}{dx}\right)^2 = \frac{(6-2x)^2}{4(6x-x^2)}$$

$$y(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2} \;=\; 3$$

$$\boxed{S = 2\pi \int_1^3 y(x)\sqrt{1+\left(\frac{dy}{dx}\right)^2}\, \mathrm{d}x = 12\pi}$$

Q3: The base of a solid is the region bounded by the curves, $y = \sqrt{x}, y = 0$, and $x = 4$. The cross-sections perpendicular to the x-axis are squares. Find the volume of this solid.

Sol(3): Side of the square, $s = \sqrt{x}$; area of the square, $A(x) = (\sqrt{x})^2 = x$.

Volume of the solid,

$$V = \int_0^4 A(x)\, \mathrm{d}x = \int_0^4 x\, \mathrm{d}x = \; 8.$$

Q4: A force of $F = \frac{x}{\sqrt{x^2+9}}$ lbs is applied to move an object along the x-axis from $x = 0$ to $x = 4$ft. Determine the amount of work done.

Sol(4): The work done,

$$W = \int_0^4 F\ \mathrm{d}x = \int_0^4 \frac{x}{\sqrt{x^2+9}}\ dx \to_{\mathrm{d}u=2x\mathrm{d}x}^{u=x^2+9} = \frac{1}{2}\int \frac{1}{\sqrt{u}}\ \mathrm{d}u = \sqrt{u} = \Big[\sqrt{x^2+9}\Big]_0^4 = (5-3) = 2.$$

Q5: The region between the curve $y = e^x + e^{-x}$, $-1 \le x \le 1$, and the x-axis is revolved around the x-axis to generate a solid. Find its volume.

Sol(5): The volume for solid or revolution for a curve is the volume of object determined by a curve $f(x)$ rotated around the x-axis on an interval $[a, b]$ given by,

$$V = \int_a^b \pi\ (f(x))^2\ \mathrm{d}x$$

$y = f(x) = e^x + e^{-x} \Rightarrow (f(x))^2 = e^{2x} + e^{-2x} + 2$

The volume,

$$V = \int_{-1}^1 \pi(e^{2x} + e^{-2x} + 2)\ \mathrm{d}x = 4\pi + \pi(e^2 - e^{-2}).$$

Q6(a): Find the derivative

$$y = x\ \ln\Big(\cosh(2x)\Big)$$

Sol(6(a)): Using the chain rule,

$$\begin{aligned}
\frac{dy}{dx} &= x\ \frac{d}{dx}\Big(\ln(\cosh(2x)\Big) + \ln\Big(\cosh(2x)\Big)\ \frac{d}{dx}(x) \\
&= x\ \frac{1}{\cosh(2x)}\ \frac{d}{dx}(\cosh(2x))\ +\ \ln\Big(\cosh(2x)\Big) \\
&= x\ \tanh(2x)\ \frac{d}{dx}(2x) + \ln\Big(\cosh(2x)\Big) \\
&= 2x\ \tanh(2x)\ +\ \ln\Big(\cosh(2x)\Big).
\end{aligned}$$

Q6(b): Evaluate the integral

$$\int_0^{\ln 2} \Big(2e^x\ \cosh(x) - \mathrm{e}^{2x}\Big)\ \mathrm{d}x$$

Sol6(b): Simply $2e^x\ \cosh(x) = e^{2x} + 1$

$$\int_0^{\ln 2} \Big(2e^x\ \cosh(x) - \mathrm{e}^{2x}\Big)\ \mathrm{d}x = \int_0^{\ln 2} e^{2x} + 1\ -\ e^{2x}\ \mathrm{d}x\ = \int_0^{\ln 2} \mathrm{d}x = \ln(2).$$

Q7. A 20 lb bucket is lifted from the ground into the air by pulling in L ft. The cable weighs 4 lb/ft Of 400 ft.lbs of work done lifting both the bucket and cable, what is the length of the cable, L?

Sol(7) The work done to lift the bucket, $W_1 = 20L$,

The work done to lift the cable, $W_2 = 4\int_0^L (L-y)\,\mathrm{d}y = 4\left[Ly - \frac{y^2}{2}\right]_0^L = 4[L^2 - \frac{L^2}{2}] = 2L^2$,

The total work done lifting both the bucket and cable, $W = 400 = W_1 + W_2$,
$\Rightarrow 400 = 20L + 2L^2 \Rightarrow (L-10)(L+20) = 0 \Rightarrow L = 10$.

Q8. A tank is constructed by revolving the curve $y = 6x^2$ for $0 \le x \le 1$ about y-axis. The tank is filled with fluid weighing 20 lb/ft^3. How much work is done in pumping all the fluid to a level 2 ft above the rim of the tank?
Sol(8) The work done in pumping all the fluid to a level 2 ft above the rim of the tank,

$$W = \int_0^6 \frac{20\pi}{6}\, y(8-y)\,\mathrm{d}y = \frac{20\pi}{6}\left[\frac{8y^2}{2} - \frac{y^3}{3}\right]_0^6 = 240\pi.$$

Q9. The region enclosed by the curves $y = x+2, y = -x+2$ and $x = 3$ is revolved about the line $x = 6$ to generate a solid. Find the volume using the shell method.
Sol(9) The volume by shell method,

$$V = \int_a^b 2\pi r h\,\mathrm{d}x$$

Here, $r = (x+2) - (-x+2) = 2x,\ h = (6-x)$
The volume,

$$V = \int_0^3 2\pi(2x)(6-x)\mathrm{d}x = 4\pi\left[\frac{6x^2}{2} - \frac{x^3}{3}\right]_0^3 = 72\pi.$$

Q. Find the volume of the solid of revolution formed by revolving the y-axis the region enclosed by

$$y = \cos(x^2)$$

and the $x-$axis.
Sol Using the Shell method, we have,

$$V = \int_0^{\sqrt{\frac{\pi}{2}}} 2\pi x\cos(x^2)\mathrm{d}x$$

(1)
$$= \pi\left[\sin(x^2)\right]_0^{\sqrt{\frac{\pi}{2}}} = \pi.$$