GENERALIZATION OF LEAPFROGGING ORBITS OF POINT VORTICES

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1. Background

Point vortex motion arises in the study of concentrated vorticity in an ideal, incompressible fluid described by Euler's equations. The two-dimensional Euler equations of fluid mechanics, a partial differential equation (PDE) system, support a solution in which the vorticity is concentrated at a single point. Helmholtz derived a system of ordinary differential equations (ODEs) that describe the motion of a set of interacting vortices that behave as discrete particles, which approximates the fluid motion in the case that the vorticity is concentrated in very small regions [6]. This system of equations has continued to provide interesting questions for over 150 years. For a thorough introduction and review see Refs [1], [17]. Kirchhoff formulated these equations as a Hamiltonian system [10]. We will discuss the special class of relative periodic orbits known as the the leapfrogging orbits. The relative periodic of a four point vortex problem two positive and two negative point vortices, all of the same absolute circulation arranged as co-axial vortex pairs is known as the leapfrogging orbit. A graphical representation of a leapfrogging orbit is shown in the Figure 1.1. This dissertation will present the generalizations to the leapfrogging motion of point vortices and vortex rings, including their

stability and dynamics in section (5.5). More specifically, we will study the leapfrogging motion of $2N$ vortices, with circulations half positive and half negative.

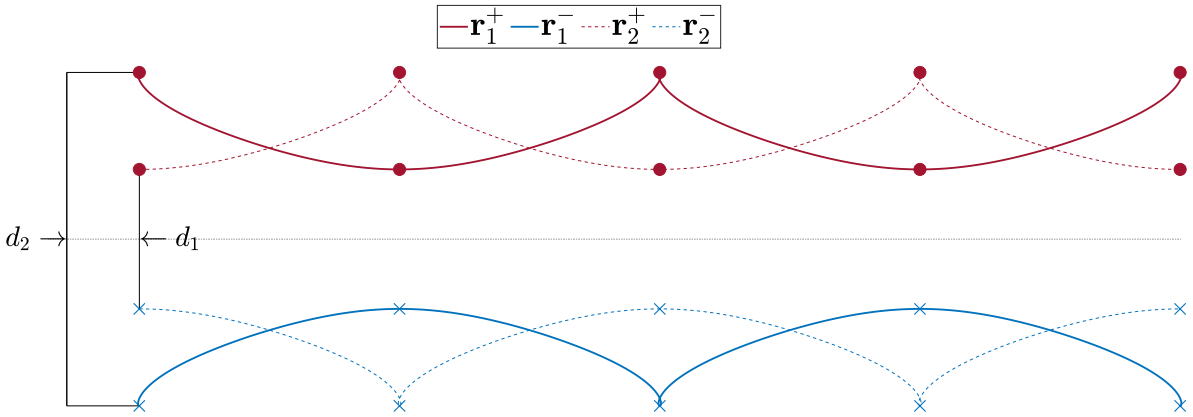

FIGURE 1.1. A leapfrogging orbit with the distances d_1 and d_2 marked, Behring[2].

2. Hamiltonian Dynamical Systems

Let $\mathcal{H}(\mathbf{q}, \mathbf{p}, t)$ be a smooth real-valued function defined on an open set of $\mathsf{R}^n \times \mathsf{R}^n \times \mathsf{R}$. The vectors $\mathbf{q}^T = (q_1, \cdots, q_n)$ and $\mathbf{p}^T = (p_1, \cdots, p_n)$ are called conjugate variables and are traditionally referred to as position and momentum vectors. A Hamiltonian system with n degrees of freedom is a system of $2n$ ODEs of the form

$$\dot{q}_i = \frac{\partial \mathcal{H}}{\partial p_i}, \quad \dot{p}_i = -\frac{\partial \mathcal{H}}{\partial q_i} \tag{2.1}$$

where $i = 1, 2, \cdots, n$. If $\mathcal{H}$ is independent of t, the system is said to be an autonomous system.

2.1. Poisson Brackets.

Poisson brackets define an algebraic structure that can be used to generalize a Hamiltonian system and is a powerful tool when working with canonical transformations. The Poisson bracket $\{\cdot, \cdot\}$ for two functions f and g is defined as

$$\{f, g\} = \sum_{i=1}^{N} \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}. \tag{2.2}$$

Let $f := f(\mathbf{q}(t), \mathbf{p}(t); t)$, then differentiating f with respect to time time, t, we get,

$$\begin{aligned} \frac{\mathrm{d}f}{\mathrm{d}t} &= \sum_{i=1}^{N} \frac{\partial f}{\partial q_i}\frac{\partial q_i}{\partial t} + \frac{\partial f}{\partial p_i}\frac{\partial p_i}{\partial t} + \frac{\partial f}{\partial t} \\ &= \sum_{i=1}^{N} \frac{\partial f}{\partial q_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q_i} + \frac{\partial f}{\partial t} \\ &= \{f, \mathcal{H}\} + \frac{\partial f}{\partial t}. \end{aligned} \tag{2.3}$$

From Equation (2.3), f is the constant of the motion or conserved quantity if and only if $\{f, \mathcal{H}\} + \frac{\partial f}{\partial t} = 0$. A function $F \in C^1(M, \mathsf{R})$ is called a **first integral** of the Hamiltonian system if $\{F, \mathcal{H}\} = 0$. M is the smooth manifold, here $M = \mathsf{R}^n$. For the coordinates q_i, p_i, we have,

$$\{q_i, q_j\} = 0, \quad \{p_i, p_j\} = 0, \text{ and } \quad \{q_i, p_j\} = \delta_{ij},$$

Another more compact and geometric formulation of a Hamiltonian system is as follows: Let $u = (\mathbf{q}, \mathbf{p}) \in \mathsf{R}^{2n}$. For each smooth function $\mathcal{H} : \mathsf{R}^{2n} \to \mathsf{R}$ the Hamiltonian vector field $X_H : \mathsf{R}^{2n} \to \mathsf{R}^{2n}$ is defined as $X_H := J\nabla\mathcal{H}(u)$, where J is the symplectic matrix defined as

$$J = \begin{pmatrix} 0 & \mathsf{I}_n \\ -\mathsf{I}_n & 0 \end{pmatrix}, \text{ where } \mathsf{I}_n \text{ is an } n \times n \text{ identity matrix.}$$

Consider the system

$$\dot{u} = X_H(u), \tag{2.4}$$

and we denote the flow of system by $\phi_H(t, u) = \phi_H^t(u)$.

Definition 1 (Involution). *Two functions f and g of the canonical phase-space coordinates (x, y) are said to be in involution if their Poisson bracket vanishes, i.e., if $\{f, g\} = 0$.*

A set of N integrals of motion that are in involution forms a set of N isolating integrals of motion.

2.2. **Integrability condition and conserved quantities.** In Hamiltonian systems, conserved quantities and symmetries are related by Noether's theorem and they result in families of non-isolated periodic orbits.

Let $\psi(t, \xi) = \psi_t(\xi)$ is the general solution of a Hamiltonian system $\dot{x} = J\nabla F(x)$, where $F : \mathsf{R}^{2n} \to \mathsf{R}$ is smooth. We say ψ_t is a symplectic symmetry for the Hamiltonian $\mathcal{H}$ if

$$\mathcal{H}(x) = \mathcal{H}(\psi(t, x)) = \mathcal{H}\left(\psi_t(x)\right) \tag{2.5}$$

for all $x \in \mathsf{R}^{2n}$ and all $t \in \mathsf{R}$.

Theorem 1 (Noether's theorem [9]). *Let ψ_t be a symplectic symmetry for the Hamiltonian $\mathcal{H}$. Then F is an integral for the Hamiltonian system with Hamiltonian $\mathcal{H}$.*

Proof. Differentiate (2.5) with respect to t to get

$$\begin{aligned} 0 &= \frac{\partial \mathcal{H}(\psi(t,x))}{\partial x} \frac{\partial \psi(t,x)}{\partial t} \\ &= \frac{\partial \mathcal{H}(\psi(t,x))}{\partial x} J \frac{\partial F(\psi(t,x))}{\partial x} \\ &= \{\mathcal{H}, F\}(\psi(t,x)). \end{aligned}$$

Setting $t = 0$ yields $\{\mathcal{H}, F\}(x) = 0$ or simply $\{\mathcal{H}, F\} = 0$. □

For the Hamiltonian systems under consideration, this allows us to draw three conclusions

- Invariance under translations in time yields that the value of the Hamiltonian is a conserved quantity (i.e., Energy is conserved).
- Invariance under translations provides conservation of momentum.
- Invariance under rotations provides conservation of angular momentum.

Theorem 2 (Liouville's Theorem). *A Hamiltonian system with n degrees of freedom which possesses n integrals of motion in involution (and thus n isolating integrals of motion) is integrable by quadrature.*

Definition 2 (Completely Integrable). *In a Hamiltonian system with n degrees of freedom ($2n$-dimensional phase space) and k mutually involutive conserved quantities, one can reduce the phase space's dimension to $2(n - k)$. If $k = n$, the system is said to be completely integrable and, in principle, can be solved by quadrature. This system is said to be completely integrable in the sense of Liouville integrability.*

3. The N-vortex Problem

Helmholtz derived the evolution equations of vorticity for Euler's model, which shows that the vorticity can neither be created or destroyed by any conservative forces. It is known today as the following theorems:

Theorem 3 (Helmholtz's first theorem). *The total vorticity ux in a vorticity tube remains constant along the tube.*

Theorem 4 (Helmholtz's second theorem). *The total vorticity ux across any material surface remains constant in time.*

By using 3 and 4, Helmholtz considered the perpendicular section of infinitely thin, straight, parallel vortex filaments with constant vorticity with a plane, thus he had introduced the point vortex model, known today as the N-vortex problem in the plane.

3.1. **The Hamiltonian N-vortex problem.** In this section, we will consider the Hamiltonian of the N-vortex problem. Setting the generalized coordinates $q_i = x_i$ and the generalized momenta $p_i = \Gamma_i y_i$ in Equation (2.1) gives the equation – in the sense of Hamiltonian dynamics – for the point vortex system.

Given a system of N vortices, each vortex $\mathbf{r}_i = (x_i, y_i)$ with intensity $\Gamma_i \in \mathsf{R}\backslash\{0\}$, their dynamics are governed by the ODEs:

$$\dot{x}_i = -\frac{1}{2\pi}\sum_{j\neq i}\frac{\Gamma_j}{|\mathbf{r}_i - \mathbf{r}_j|^2}(y_i - y_j), \qquad \dot{y}_i = \frac{1}{2\pi}\sum_{j\neq i}\frac{\Gamma_j}{|\mathbf{r}_i - \mathbf{r}_j|^2}(x_i - x_j). \tag{3.1}$$

Kirchhoff [10] established the Hamiltonian representation of these equations of motion as a nonlinear coupled system of $2N$ ordinary differential equations:

$$\Gamma_i \frac{\mathrm{d}x_i}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial y_i}, \qquad \Gamma_i \frac{\mathrm{d}y_i}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial x_i}, \tag{3.2}$$

and the system of ODEs describing the N-vortex motion can be described by the Hamiltonian [17],

$$\mathcal{H}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \cdots, \mathbf{r}_N) = -\frac{1}{4\pi}\sum_{1\ \ i<j\ \ N}\Gamma_i\Gamma_j\ \log\|\mathbf{r}_i - \mathbf{r}_j\|^2 \tag{3.3}$$

is conserved during the motion of the point vortices. The above formulation uses a slightly nonstandard formulation of the Poisson bracket in terms of circulation strength by setting $q_i = x_i$ and $p_i = \Gamma_i y_i$ in the Equation (2.3), to get

$$\{F, \mathcal{H}\} = \sum_{i=1}^{N}\frac{1}{\Gamma_i}\left(\frac{\partial F}{\partial x_i}\frac{\partial \mathcal{H}}{\partial y_i} - \frac{\partial F}{\partial x_i}\frac{\partial \mathcal{H}}{\partial y_i}\right). \tag{3.4}$$

In addition to the the Hamiltonian (3.3), $\mathcal{H}$, has three independent first integrals:

$$Q = \sum_{i=1}^{N}\Gamma_i x_i, \qquad P = \sum_{i=1}^{N}\Gamma_i y_i, \text{ and } \quad I = \sum_{i=1}^{N}\Gamma_i(x_i^2 + y_i^2). \tag{3.5}$$

The two components of the linear impulse, Q and P, result from the translational invariance. The angular impulse I is a constant of the motion because of the rotational invariance of $\mathcal{H}$, Equation (3.5). The presence of conserved quantities allows one to reduce the effective number of degrees of freedom. The so-called fundamental Poisson brackets are

$$\{x_i, y_j\} = \frac{\delta_{ij}}{\Gamma_i}, \tag{3.6}$$

and

$$\{x_i, x_j\} = 0, \quad \{y_i, y_j\} = 0. \tag{3.7}$$

From direct calculations from Equations (3.4) and (3.5), we get,

$$\{Q, \mathcal{H}\} = \{P, \mathcal{H}\} = 0, \quad \{I, \mathcal{H}\} = 0. \tag{3.8}$$

Note: $Q, P,$ and I do not depend explicitly on time, it follows once again that $Q, P,$ and I are integrals of the motion.

4. N-VORTEX MOTION

4.1. **One-vortex motion.** For the one-point vortex with the intensity Γ_1 at $\mathbf{r_1} = (x_1, y_1)$, the Hamiltonian from (3.3) is

$$\mathcal{H}(\mathbf{r_1}) = 0, \tag{4.1}$$

which shows $\mathcal{H}$ is time-independent. Hence, the vortex does not move.

Example 4.1. *Consider the motion of a particle positioned at (x, y) in the flow field of a particle moving in the flow due to an isolated point vortex of strength Γ located at the origin. The Hamiltonian is given by*

$$\mathcal{H}(\mathbf{r}) = -\frac{\Gamma}{4\pi}\log\|\mathbf{r}\| = -\frac{\Gamma}{4\pi}\log\left(\sqrt{x^2+y^2}\right), \tag{4.2}$$

with the equations of motion from (3.1),

$$\dot{x} = -\frac{\Gamma}{2\pi}\frac{y}{\sqrt{x^2+y^2}}, \quad \dot{y} = \frac{\Gamma}{2\pi}\frac{x}{\sqrt{x^2+y^2}}. \tag{4.3}$$

Making a canonical transformation of

$$x = \sqrt{2r}\cos(\theta), \quad y = \sqrt{2r}\sin(\theta),$$

(4.3) reduces to

$$\dot{r} = 0, \quad \dot{\theta} = \frac{\Gamma}{2\pi\sqrt{2r}},$$

so that r is a constant of motion.

4.2. **Two-vortex Motion.** In this section, we demonstrate that two vortices of equal and opposite-signed vorticity move in parallel at a uniform speed with their common velocity inversely proportional to the distance between them. Two vortices of equal and like-signed vorticity, by contrast, trace a circular path with a constant rotation rate proportional to the inverse square of the distance between them. See Figure 4.1 for both cases.

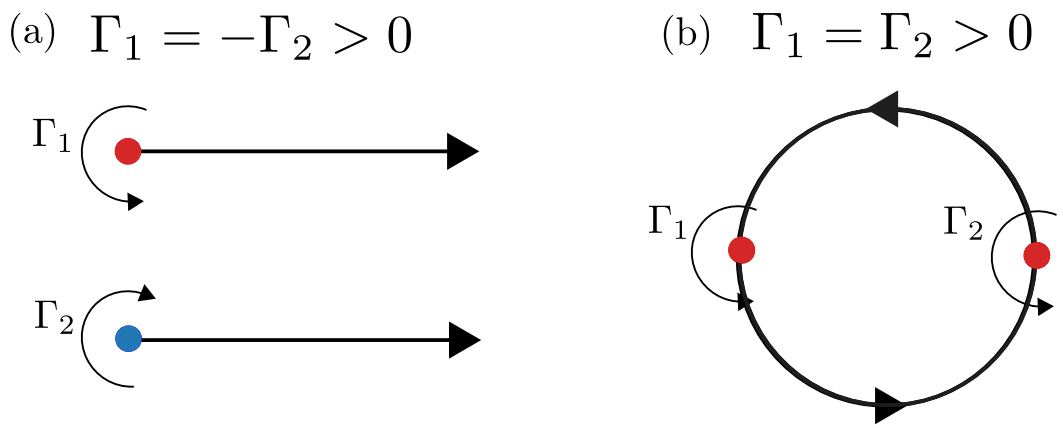

FIGURE 4.1. (a) Opposite-signed vortices move in parallel along straight lines. (b) Like-signed vortices move along a circular path, Behring [2].

Consider a vortex with the circulation Γ_1 at $\mathbf{r}_1 = (x_1, y_1)$ and a second vortex with circulation Γ_2 at $\mathbf{r}_2 = (x_2, y_2)$, the Hamiltonian for the system using (3.3) is,

$$\mathcal{H}(\mathbf{r}_1, \mathbf{r}_2) = -\frac{1}{4\pi}\Gamma_1\Gamma_2\log\|\mathbf{r}_1 - \mathbf{r}_2\|^2. \tag{4.4}$$

The behavior depends on whether the net circulation is zero, so each case will be considered separately.

Case I: $\Gamma_1 + \Gamma_2 \neq 0$. After noting that $\mathcal{H}$ depends on the coordinates only, we define the canonical transformation to center-of-vorticity and coordinates difference as

$$\mathbf{c} = \frac{\Gamma_1 \mathbf{r}_1 + \Gamma_2 \mathbf{r}_2}{\Gamma_1 + \Gamma_2}, \quad \mathbf{d} = \mathbf{r}_1 - \mathbf{r}_2, \quad D = \|\mathbf{d}\|. \tag{4.5}$$

More explicitly,

$$\mathbf{r_1} = \mathbf{c} + \frac{\Gamma_2}{\Gamma_1 + \Gamma_2}\mathbf{d} \tag{4.6}$$

$$\mathbf{r_2} = \mathbf{c} - \frac{\Gamma_2}{\Gamma_1 + \Gamma_2}\mathbf{d} \tag{4.7}$$

For the system of two vortices, the equations of motion are:

$$\dot{x}_1 = -\frac{1}{2\pi}\frac{\Gamma_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2}(y_1 - y_2), \quad \dot{x}_2 = -\frac{1}{2\pi}\frac{\Gamma_1}{|\mathbf{r}_1 - \mathbf{r}_2|^2}(y_1 - y_2), \tag{4.8}$$

and

$$\dot{y}_1 = \frac{1}{2\pi}\frac{\Gamma_2}{|\mathbf{r}_1 - \mathbf{r}_2|^2}(x_1 - x_2), \quad \dot{y}_2 = \frac{1}{2\pi}\frac{\Gamma_1}{|\mathbf{r}_1 - \mathbf{r}_2|^2}(x_1 - x_2). \tag{4.9}$$

Let

$$\boxed{X \equiv x_1 - x_2, \quad Y \equiv y_1 - y_2}.$$

For X and Y, the equations of motion are,

$$\dot{X} = -\frac{\omega}{2\pi}Y, \quad \dot{Y} = \frac{\omega}{2\pi}X, \tag{4.10}$$

where

$$D^2 = X^2 + Y^2, \quad \omega = \frac{\Gamma_1 + \Gamma_2}{D^2}. \tag{4.11}$$

Now,

$$\frac{\mathrm{d}D^2}{\mathrm{d}t} = 0, \text{ which proves that } D^2 \text{ is a constant of motion.} \tag{4.12}$$

The two-degree-of-freedom Hamiltonian (4.4) is now reduced to the one degree of freedom in $\mathbf{d}$, since D is a constant of motion,

$$H(\mathbf{d}) = -\Gamma_1\Gamma_2 \log \|\mathbf{d}\|^2.$$

From 1, D is an integral for the Hamiltonian system (4.4).

WLOG, let $\Gamma_1 = \Gamma_2 = 1$, then from Equations (4.6) and (4.7), we have,

$$\mathbf{r_1} - \mathbf{c} = \frac{1}{2}\mathbf{d}, \tag{4.13}$$

$$\mathbf{r_2} - \mathbf{c} = -\frac{1}{2}\mathbf{d}, \tag{4.14}$$

and the vortices located at $\mathbf{r_1}$ and $\mathbf{r_2}$ rotate at constant speed at opposite ends of a diameter of a circle.

Also, $\mathbf{c} = \frac{1}{2}(\mathbf{r_1} + \mathbf{r_2})$, the center of vorticity is a constant of motion using Equations (4.8) and (4.9).

Case II: $\Gamma_1 = -\Gamma_2$. The center-of-vorticity in Equation (4.5) is not defined when the net circulation is zero. The coordinates of the linear impulse $\mathbf{A} = (Q, P) = \sum \Gamma_i \mathbf{r}_i$ are independently conserved.

For the linear impulse to lie in y direction we set $Q = 0, P = D$, and without loss of generality, the vorticity $\Gamma_1 = -\Gamma_2 = 1$. The new linear impulse is $\mathbf{A} = \mathbf{d} = (0, D)$. That is, $x_1 = x_2$, and $y_1 = y_2 + D$. The equation of motion (3.1) gives $\dot{x}_1 = \dot{x}_2 = \frac{1}{D}$ and $\dot{y}_1 = \dot{y}_2 = 0$. That is, the vortices move in parallel along the x-axis with a constant speed of $\frac{1}{D}$.

4.3. **Three-vortex Motion.** In this section, we will discuss about the integrability for $N = 3$ point vortices. Using Equation (3.5), and taking the Poisson bracket of $Q, P,$ and I with one another, we have,

$$\{Q, P\} = \sum_{i=1}^{N} \Gamma_i, \quad \{Q, I\} = 2P, \quad \{P, I\} = -2Q. \tag{4.15}$$

Also,

$$\begin{aligned} \{Q^2 + P^2, I\} &= Q\{Q, I\} + \{Q, I\}Q + P\{P, I\} + \{P, I\}P \\ &= 2Q\{Q, I\} + 2P\{P, I\} \\ &= 4PQ - 4PQ = 0. \end{aligned} \tag{4.16}$$

It can be easily verified that,

$$\{\mathcal{H}, I\} = 0, \{\mathcal{H}, Q^2 + P^2\} = 0. \tag{4.17}$$

From Equations (4.16) and (4.17), we have three independent integrals in involution, $\mathcal{H}, I,$ and $Q^2 + P^2$. Thus, 2 assures the integrability of $N = 3$ vortex problem regardless of the vortex strengths [1].

5. Leapfrogging Orbits

5.1. **Introduction.** Leapfrogging is a relative periodic orbit of the four-vortex problem with two positive and two negative point vortices, all of the same absolute circulation arranged as co-axial vortex pairs. First described by Gröbli in 1877 [5] and independently by Love [13]. The leapfrogging motion of vortex rings was first described by Helmholtz [6].

(1) Opposite-signed vortices move in parallel along straight lines.
(2) Like-signed vortices move in a circular path with a constant rotation rate.

5.2. **Vortex pair leapfrogging.** Considering the intersection of a pair of coaxial vortex rings with a plane lying along one of their common diameters, we find four point vortices: two with positive vorticity and two with negative. Therefore, we can consider the motion of four vortices, two with circulation $+1$ and two with -1 as a simplified model for the leapfrogging of two vortex rings. For a general system of N vortices in the plane with positions $\mathbf{r}_i$ and circulations Γ_i, the Hamiltonian takes the form Equation (3.3). At $t = 0$, the four vortices are arranged collinearly spaced with with distance d_1 and d_2 as shown in Figure 1.1.

Twice per period, the vortices pass through a collinear state, and the existence of the periodic orbit depends on a ratio $\alpha = \frac{d_1}{d_2}$ as shown in Figure 1.1. Love showed that periodic leapfrogging orbits exist for $\alpha > \alpha_{\text{exist}} = 3 - 2\sqrt{2}$ of oppositely-signed vortices, albeit with doubled vorticity. The inner vortices start moving outwards and the outer vortices start moving inwards, loses collinearity for some period of time. After some time moving in the same direction, the vortices are collinear again. At this point, the inner vortices are now the outer vortices, and the outer vortices are the inner vortices.

As $\alpha \to 1$, each like signed pair coalesces and the motion degenerates into the familiar straight-line motion of a pair of oppositely-signed vortices, albeit with doubled vorticity as shown in Figure 5.1.

In the next section, we will discuss the concept of relative equilibria and relative periodic orbits mathematically. To describe it mathematically, we need to a few basic definitions.

5.3. **Relative Equilibria.** Group Action: If G is a group with identity element e, and X is a set, then a **(left) group action** α of G on X is a function

$$\alpha : G \times X \to X$$

that satisfies the following two axioms:

- **Existence of an identity element :** For each $x \in X, \exists!\, e \in G$ such that $\alpha(e, x) = e * x = x$
- **Group Property:** $\forall g, h \in H$ and $x \in X$, $\alpha(g, \alpha(h, x)) = g * (h * x) = (gh) * x = \alpha(gh, x)$.

The group G is said to act on X (**from the left**). A set X together with an action of G is called a (left) G-set. $(G, *)$ is a group with the operation$(*)$ on it.
A **continuous dynamical system** is a continuous map

$$\phi : (\mathsf{R}, +) \times M \to M$$

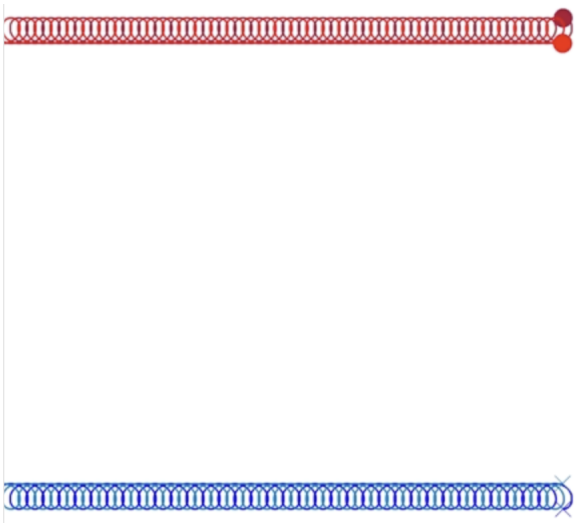

FIGURE 5.1. Vortex pair with double vorticity.

and the map is a group action. $(\mathsf{R}, +)$ denotes the additive group of real numbers whose identity element $e = 0$. That is, $\phi_t(x) = \phi(t, x)$ satisfies the following properties:

- $\phi(0, x) = \phi_0(x) = x$, for all $x \in M$, and
- $\forall t, s \in \mathsf{R}$,

$$\phi_t \circ \phi_s = \phi_{t+s}$$

where the composition symbol, $\circ$, means, $(\phi_t \circ \phi_s)(x) = \phi_t(\phi_s(x))$.

Definition 3 (Equivariant Map)**.** *Let M and N be smooth manifolds with smooth G-action. A smooth map $f : M \to N$ is called equivariant if it commutes with the given G-actions, i.e.*

$$f(g \cdot m) = g \cdot f(m), \forall g \in G \text{ and } \forall m \in M.$$

5.4. **Relative Equilibria in N-vortex problems.**

Definition 4. *An orbit $\{\phi_H(t, p_0) : t \in \mathsf{R}\}$ of the system (2.4) is a relative equilibrium (with respect to a rotation group $\mathcal{R}$) if there exists a mapping $\gamma : \mathsf{R} \to \mathcal{R}$ such that $\phi_H(t, p_0) = \gamma(t)(p_0)$.*

For the rotation group or the translation group the relative equilibria correspond to equilibria in a uniformly rotating frame. The solution to the case II in section 4.2 is an example of such a relative equilibrium.

5.5. **Generalization of Leapfrogging.** In section 4.2, we saw that if $\Gamma_1 = \Gamma_2$, then the two vortices rotate about each other rigidly. In this section, we will discuss the rigidly-rotating cluster of vortices. Figure 5.2 shows that leapfrogging orbit in a limit of large limit, consists of one such pair and its mirror image at a large separation, uniformly translating along their axis of symmetry. Previous groups have studied the generalization of two mirror image nearly-rigidly rotating clusters. Wacks et al. [19] considered both rings point vortices. The previous studies just solved some initial value problems. They did not systematically construct to continue the relative periodic orbits nor consider their linear stability. An older paper by Konstantinov [11] considers the leapfrogging of N-vortex rings, and shows via post-hoc analysis of numerical simulations that motion of three or more vortex rings is in general chaotic, though he does not explicitly construct the periodic orbits nor analyze their stability. The research on Leapfrogging in systems of $N \geq 3$ (with opposite pairs of vorticity) coaxial vortex rings does not explicitly gives any information about the construction of periodic orbits nor analyze their stability. In the next section, we will generalize the leapfrogging problem with the rigidly-rotating cluster of vortices. Consider any rigidly-rotating cluster of vortices, shown in Figure 5.2 (a-e), and using the notation of subfigures (d-e), place one such cluster of radius a at a distance R above the x-axis and its mirror image, consisting of vortices of opposite circulation, at a distance R below. The distance between the two clusters is $2R$.

The limit $\beta = \frac{a}{R} \ll 1$, will definitely leapfrog, as the velocity field at sufficiently large distance from each cluster will approach that of a point vortex. The question remains what happens to the leapfrogging motion as we decrease the distance between the two clusters. The idea to compute these families of leapfrogging orbits have never been explored numerically. We propose to perform a full numerical continuation study to deal this situation.

6. Numerical Methods

The role of periodic solutions in Hamiltonian systems and their importance in modern physics was first recognized by Poincaré [18]. It is well known that the bifurcation of periodic orbits constitutes the backbone of a Hamiltonian system. If we succeed in analyzing the branching and stability behavior of these solutions as the parameters or the energy are varied, we can hope to understand and predict the general evolution of systems with symmetries and conserved quantities. However, Hamiltonian systems are non-generic dynamical systems with remarkable properties, in particular with respect to periodic orbits.

It is well known that general (dissipative) and Hamiltonian (or more generally conservative) systems behave quite differently with respect to periodic orbits, their continuation and their bifurcations. In dissipative systems periodic orbits are generically isolated (limit cycles), and therefore an external parameter is required in order to be able to continue such periodic orbits. Hamiltonian systems are special when it comes to the continuation of periodic orbits. The cylinder theorem can be used to continue the periodic orbits, which says that periodic orbits appear in one- or more-parameter families and that under appropriate non-degeneracy conditions, these families are persistent under small Hamiltonian perturbations.

In Hamiltonian systems, a common approach is to use the conserved quantity to reduce the dimension of the system. The approach which we propose here increases the dimension rather than reducing it, and reformulates the problem in a form where boundary value continuation methods can be applied directly. We discuss the methods and techniques we use in the continuation of periodic orbits in the next section.

6.1. Numerical methods for the continuation of fixed points.

Continuation schemes are used to determine how solutions of the system

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}; \mathbf{M}), \mathbf{x} \in \mathsf{R}^n, \mathbf{M} \in \mathsf{R}^m, \mathbf{F} : \mathsf{R}^n \times \mathsf{R}^m \to \mathsf{R}^n \tag{6.1}$$

vary with a certain parameter, here $\mathbf{M}$. The basic continuation algorithms are for the computation of one-dimensional solution manifolds called solution branches. For simplicity, let $\mathbf{M} = \alpha \in \mathsf{R}$.

6.1.1. *Parameter Continuation.* In this scheme, the parameter α is used as a continuation parameter of the system (6.1). Given solution $(\mathbf{x_0}, \alpha_0)$ of the system (6.1), the Implicit Function Theorem guarantees that under suitable genericity conditions there exists a unique solution branch that passes through $(\mathbf{x_0}, \alpha_0)$. We aim to find another point say, $(\mathbf{x_1}, \alpha_1)$ on this branch. To do that, one can use a root-finding technique,

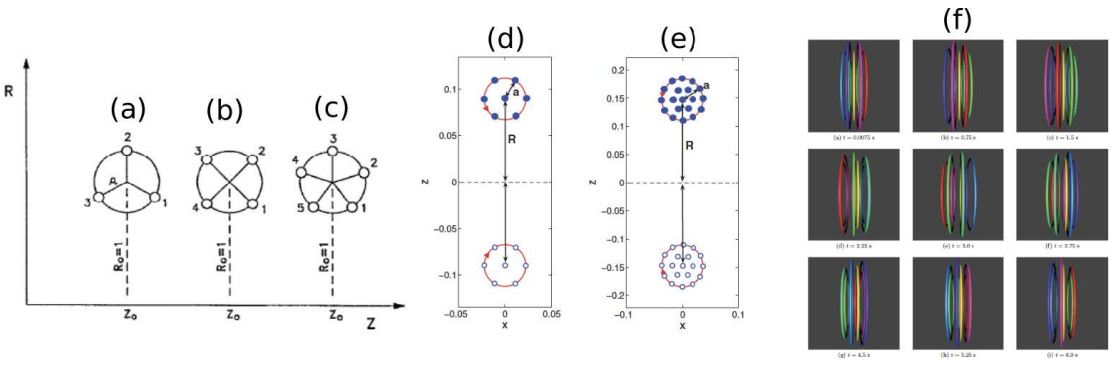

Figure 5.2. (**a-c**) The near-leapfrog initial conditions studied numerically by Konstantinov. (**d-e**) The near-leapfrog initial conditions studied numerically by Wacks et al. (**f**) A leapfrogging-like motion of 7 vortex rings.

such as Newton's method, to solve the system

$$\mathbf{F}(\mathbf{x}_1, \alpha_1) = 0, \text{ where } \alpha_1 = \alpha_0 + \Delta\alpha. \tag{6.2}$$

To solve the Equation (6.2) for $\mathbf{x}_1$, we use **Newton's method**, to get,

$$\mathbf{F}_{\mathbf{x}}(\mathbf{x}_1^{(n)}, \alpha_1)\Delta\mathbf{x}_1^{(n)} = -\mathbf{F}(\mathbf{x}_1^{(n)}, \alpha_1), \quad \mathbf{x}_1^{(n+1)} = \mathbf{x}_1^{(n)} + \Delta\mathbf{x}_1^{(n)}, \tag{6.3}$$

for $n = 0, 1, 2, 3, \cdots$. As initial approximation use the linearization

$$\mathbf{x}_1^{(0)} = \mathbf{x}_0 + \Delta\alpha\frac{\partial \mathbf{F}}{\partial \alpha}(\mathbf{x}_0, \alpha_0).$$

If

$$\mathbf{F}_{\mathbf{x}}(\mathbf{x}_1, \alpha_1) \text{ is non singular,} \tag{6.4}$$

and Δ sufficiently small, then this iteration will converge. We repeat the process to find $\mathbf{x}_2, \mathbf{x}_3, \cdots$. The parameter continuation scheme is bound to break down at turning points (or folds), where $\mathbf{F}_{\mathbf{x}}$ is singular. To overcome this problem, we use **pseudo-arclength continuation scheme** proposed by Keller [7],[8], which we discuss in the next section.

6.2. **Pseudo-arclength Continuation.** The Pseudo-arclength continuation scheme allows continuation of a solution family past a fold. In this case, $\mathbf{x}$ and α are considered to be functions of s; that is, $\mathbf{x} = \mathbf{x}(s)$ and $\alpha = \alpha(s), \alpha \in \mathsf{R}$. Now, we seek $\mathbf{x}$ and α such that

$$\mathbf{F}(\mathbf{x}(s), \alpha(s)) = 0. \tag{6.5}$$

Differentiating (6.5) with respect to s gives

$$\mathbf{F}_{\mathbf{x}}(\mathbf{x}, \alpha)\dot{\mathbf{x}} + \mathbf{F}_{\alpha}(\mathbf{x}, \alpha)\dot{\alpha} = 0, \tag{6.6}$$

where $\dot{\mathbf{x}} = \frac{\mathrm{d}\mathbf{x}}{\mathrm{d}s}$ and $\dot{\alpha} = \frac{\mathrm{d}\alpha}{\mathrm{d}s}$.
The system (6.6) consists of n linear algebraic equations in $(n+1)$ unknowns $\dot{\mathbf{x}}$ and $\dot{\alpha}$. To determine the unknowns uniquely, we need an additional condition, which is the Euclidean arclength normalization, given as

$$\dot{\mathbf{x}}^{\mathsf{T}}\dot{\mathbf{x}} + \dot{\alpha}^2 = 1, \text{ where } \mathsf{T} \text{ is the matrix transpose.} \tag{6.7}$$

The initial conditions for Equations (6.6) and (6.7) are given by

$$\mathbf{x} = \mathbf{x}_0 \text{ and } \alpha = \alpha_0 \text{ at } s = 0. \tag{6.8}$$

Solving Equations (6.6) and (6.7) exactly would constitute arclength continuation, but the non-linearity of Equation (6.7) makes this difficult. Instead of using the Equation (6.7), pseudo-arclegth continuation constructs the next point on the curve as follows. Given a solution $(\mathbf{x}_0, \alpha_0)$ of the system (6.5), as well as the normalized direction vector $(\dot{\mathbf{x}}_0, \dot{\alpha_0})$ [See Figure 6.1].

$$\text{Let } (\tilde{\mathbf{x}}, \tilde{\alpha}) := (\mathbf{x}_0, \alpha_0) + \Delta s(\dot{\mathbf{x}}_0, \dot{\alpha}_0).$$

Then to find $(\mathbf{x}_1, \alpha_1)$ such that,

$$\langle(\mathbf{x}_1 - \tilde{\mathbf{x}}, \alpha_1 - \tilde{\alpha})|(\mathbf{x}_0 - \tilde{\mathbf{x}}, \alpha_0 - \tilde{\alpha})\rangle = 0. \tag{6.9}$$

Equations (6.7) and (6.9) give us

$$\mathbf{F}(\mathbf{x}_1, \alpha_1) = 0, \text{ and } (\mathbf{x}_1 - \mathbf{x}_0)^{\mathsf{T}}\dot{\mathbf{x}}_0 + (\alpha_1 - \alpha_0)\dot{\alpha}_0 - \Delta s = 0. \tag{6.10}$$

Using Newton's method to solve the system (6.10), we get

$$\begin{bmatrix} \left(\mathbf{F}_{\mathbf{x}}^1\right)^{(n)} & \left(\mathbf{F}_{\alpha}^1\right)^{(n)} \\ \dot{\mathbf{x}}_0^{\mathsf{T}} & \dot{\alpha_0} \end{bmatrix} \begin{bmatrix} \Delta\mathbf{x}_1^{(n)} \\ \Delta\alpha_1^{(n)} \end{bmatrix} = -\begin{bmatrix} \mathbf{F}(\Delta\mathbf{x}_1^{(n)}, \Delta\alpha_1^{(n)}) \\ \left(\mathbf{x}_1^{(n)} - \mathbf{x}_0\right)^{\mathsf{T}}\dot{\mathbf{x}}_0 + \left(\alpha_1^{(n)} - \alpha_0\right)\dot{\alpha}_0 - \Delta s \end{bmatrix}.$$

We can solve the next direction vector by solving,

$$\begin{bmatrix} \mathbf{F}_{\mathbf{x}}^1 & \mathbf{F}_{\alpha}^1 \\ \dot{\mathbf{x}}_0^{\mathsf{T}} & \dot{\alpha_0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}_1 \\ \dot{\alpha}_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and normalize the solution to keep the arclength condition valid.

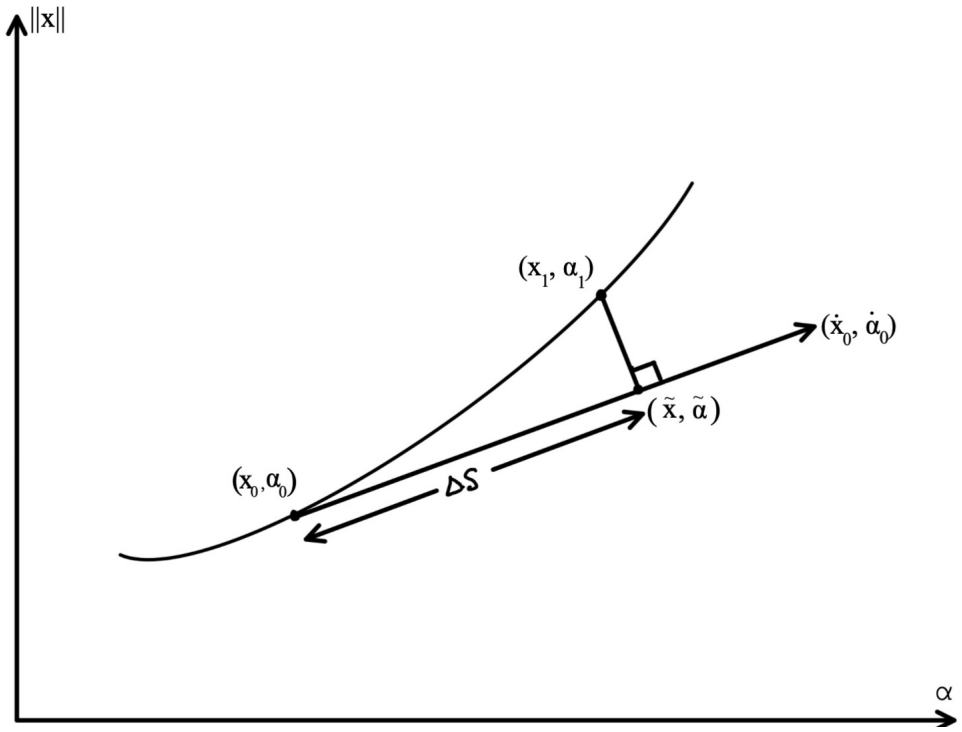

FIGURE 6.1. Graphical interpretation of pseudo-arclength continuation.

6.3. **Shooting Method For Periodic Orbits.** Our interest is not in continuing branches of equilibria, but branches of relative periodic orbits. Before we can discuss such continuation, we need to discuss how to find a single periodic orbit. In this section, we will discuss the use of the shooting method for finding the periodic solutions of Equation (6.1). To do that, we convert the initial value problem into a two-point boundary value problem. That is, our goal is to seek an initial condition $\mathbf{x}(0) = \boldsymbol{\eta}$ and a solution $\mathbf{x}(t;\boldsymbol{\eta})$ with a minimal period T such that

$$\mathbf{x}(T, \boldsymbol{\eta}) = \boldsymbol{\eta}. \tag{6.11}$$

Now, Equations (6.1) and (6.11) represent the two-point boundary value problem. The trajectory of the solution runs from $\boldsymbol{\eta}$ at $t = 0$ and returns to the same location at $t = T$, which represents the periodic solution. Let $(T_0, \boldsymbol{\eta}_0)$ be the initial guess for $(T, \boldsymbol{\eta})$. This initial guess need not be accurate enough and might need some correction. See Figure 6.2.

To correct our initial guess, we use the Newton-Raphson scheme described below. The idea is to make T close to T_0 and $\boldsymbol{\eta}$ close to $\boldsymbol{\eta}_0$, such that (6.11) is satisfied. We set

$$\delta\boldsymbol{\eta} = \boldsymbol{\eta} - \boldsymbol{\eta}_0, \quad \delta T = T - T_0 \tag{6.12}$$

such that (6.11) is satisfied; that is

$$\mathbf{x}(T_0 + \delta T, \boldsymbol{\eta}_0 + \delta\boldsymbol{\eta}) - (\boldsymbol{\eta}_0 + \delta\boldsymbol{\eta}) \simeq 0. \tag{6.13}$$

Linearizing (6.13) in terms of $\delta\boldsymbol{\eta}$ and δT, we get

$$\left[\frac{\partial \mathbf{x}}{\partial \boldsymbol{\eta}}(T_0, \boldsymbol{\eta}_0) - \mathbf{I}\right]\delta\boldsymbol{\eta} + \left[\frac{\partial \mathbf{x}}{\partial T}(T_0, \boldsymbol{\eta}_0)\right]\delta T = \boldsymbol{\eta}_0 - \mathbf{x}(T_0, \boldsymbol{\eta}_0) \tag{6.14}$$

where $\frac{\partial \mathbf{x}}{\partial \boldsymbol{\eta}}$ is an $n \times n$ matrix, $\mathbf{I}$ is an $n \times n$ identity matrix, and $\frac{\partial \mathbf{x}}{\partial T}$ is an $n \times 1$ vector. From Equation (6.1), we have

$$\frac{\partial \mathbf{x}}{\partial T}(T_0, \boldsymbol{\eta}_0) = \mathbf{F}(\boldsymbol{\eta}_0; \mathbf{M}). \tag{6.15}$$

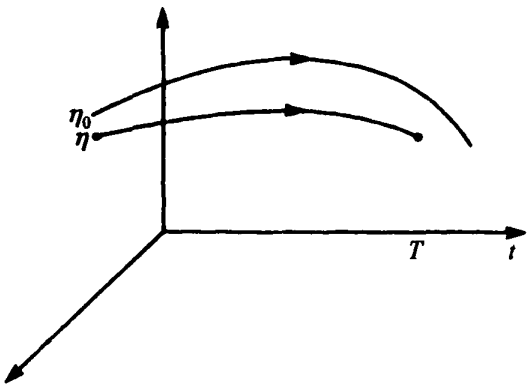

FIGURE 6.2. Depiction of the shooting method for a two-dimensional autonomous system, Nayfeh [16].

We need to determine the matrix $\frac{\partial \mathbf{x}}{\partial \boldsymbol{\eta}}$ at $(T_0, \boldsymbol{\eta}_0)$ from (6.14). Let $\mathbf{F} \in \mathcal{C}^r, r \geq 1$. Differentiating both sides of (6.1) with respect to $\boldsymbol{\eta}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \mathbf{x}}{\partial \boldsymbol{\eta}}\right) = D_{\mathbf{x}}\mathbf{F}(\mathbf{x}; \mathbf{M})\frac{\partial \mathbf{x}}{\partial \boldsymbol{\eta}} \tag{6.16}$$

with the initial condition

$$\frac{\partial \mathbf{x}}{\partial \boldsymbol{\eta}}(0) = \mathbf{I}. \tag{6.17}$$

Thus, once $\mathbf{x}$ is determined, we can solve the initial value problem (6.16) and (6.17) from 0 to T_0, and find $\frac{\partial \mathbf{x}}{\partial \boldsymbol{\eta}}$ at $(T_0, \boldsymbol{\eta}_0)$. Once $\frac{\partial \mathbf{x}}{\partial \boldsymbol{\eta}}$ at $(T_0, \boldsymbol{\eta}_0)$ is known, from (6.14) we have a system of n equations in $n+1$ unknowns, $\delta\boldsymbol{\eta}$ and δT. Therefore, to obtain a unique solution to the system (6.14), we need an additional condition. We use an additional orthogonality condition introduced Mees [14], in which we assume the corrections of $\delta\boldsymbol{\eta}$ to be orthogonal to the vector $\mathbf{F}$. That is,

$$\mathbf{F}^{\mathsf{T}}\delta\boldsymbol{\eta} = 0. \tag{6.18}$$

Using (6.18), the system (6.14) can now be represented as:

$$\begin{bmatrix} \frac{\partial \mathbf{x}}{\partial \boldsymbol{\eta}} - \mathbf{I} & \mathbf{F}(\boldsymbol{\eta}_0; \mathbf{M}) \\ \mathbf{F}^{\mathsf{T}}(\boldsymbol{\eta}_0; \mathbf{M}) & 0 \end{bmatrix} \begin{bmatrix} \delta\boldsymbol{\eta} \\ \delta T \end{bmatrix} = \begin{bmatrix} \boldsymbol{\eta}_0 - \mathbf{x}(T_0, \boldsymbol{\eta}_0) \\ 0 \end{bmatrix}.$$

Solving the above system, we will obtain the corrected values for $\delta\boldsymbol{\eta}$ and δT. We can introduce the tolerances, δ_1, δ_2, to define the convergence criteria. The convergence criterion is that the condition

$$\|\delta\boldsymbol{\eta}\| < \delta_1, \quad |\delta T| < \delta_2 \tag{6.19}$$

holds. We keep updating our initial guess $(T_0, \boldsymbol{\eta}_0)$ until the condition (6.19) holds within the specified tolerance. As a by-product of the shooting technique, one can obtain the monodromy matrix from

$$\mathcal{M} = \frac{\partial \mathbf{x}}{\partial \boldsymbol{\eta}}(T, \boldsymbol{\eta}) \tag{6.20}$$

and hence determine the stability of the calculated periodic solution by examining the eigenvalues of $\mathcal{M}$. We implemented the shooting method and used MATLAB to generate the periodic orbits for the Lorenz and Rössler system, which are shown in the Figures 6.3 and 6.4 respectively.

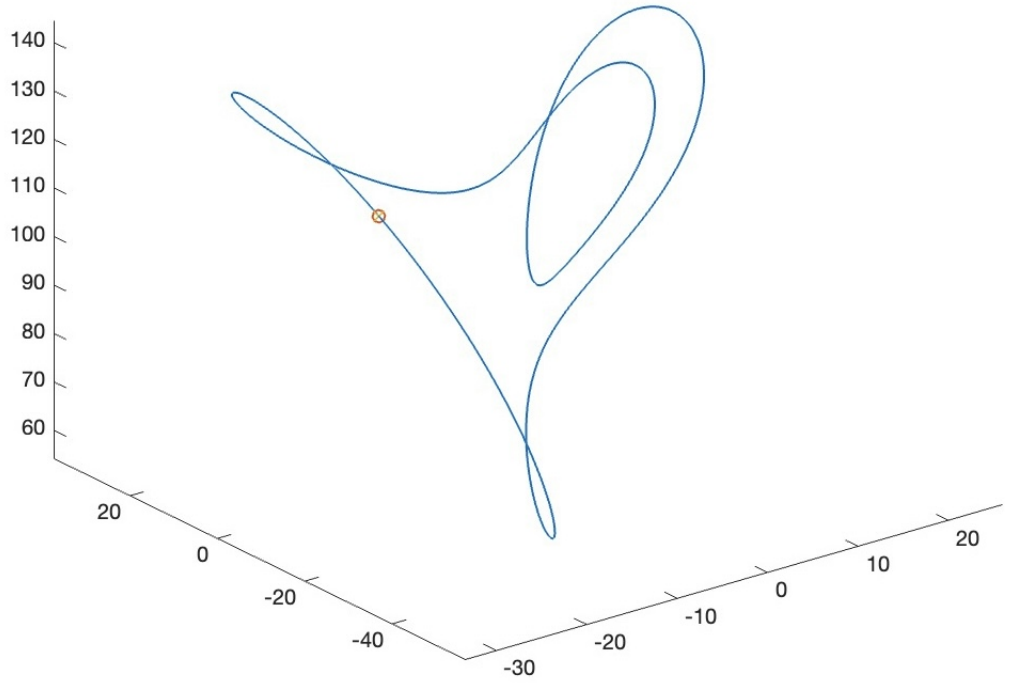

FIGURE 6.3. Periodic Orbit of the Lorenz System.

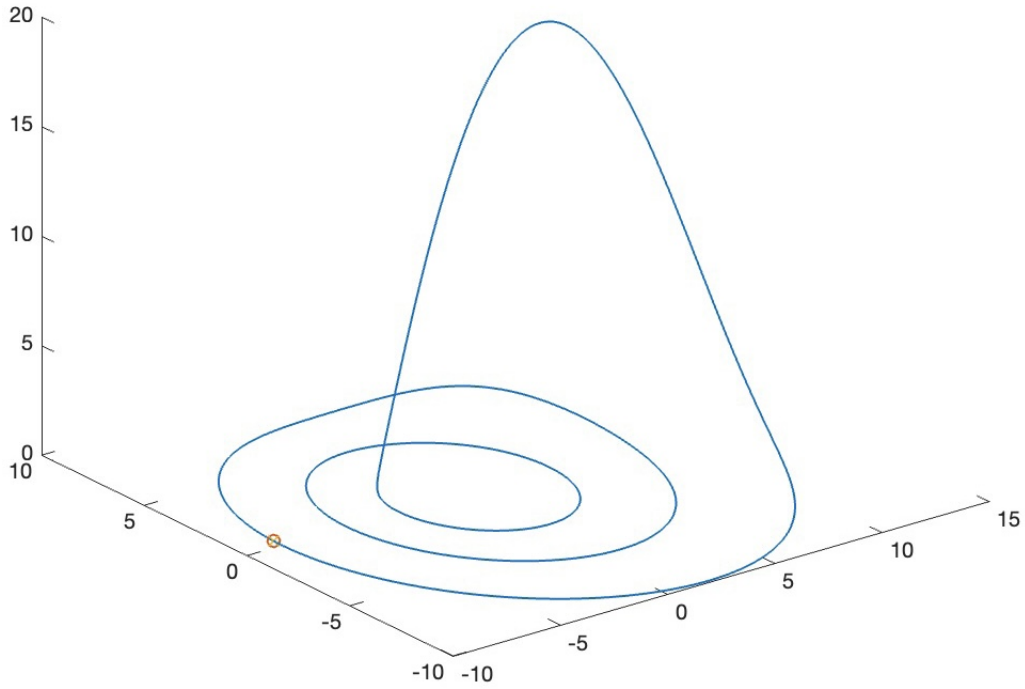

FIGURE 6.4. Periodic Orbit of the Rössler System.

6.3.1. *Reformulation of the system* (6.1) *to period one.* For a periodic solution of an autonomous system, the period T is an unknown quantity, and it is determined along with the states. The two-point boundary-value

problem is given by (6.1) and (6.11). For convenience, one may change independent, variable t in (6.1) to a new independent variable T such that $t \to \frac{t}{T}$, where T is the period of the solution. Then, Equations (6.1) and (6.11) become

$$\dot{\mathbf{x}} = T\ \mathbf{F}(\mathbf{x}; \mathbf{M}) \tag{6.21}$$

$$\mathbf{x}(1; \boldsymbol{\eta}, T) = \boldsymbol{\eta} \tag{6.22}$$

where $\mathbf{x}(1; \boldsymbol{\eta}, T) = \boldsymbol{\eta}$ has the unit period. Thus, the problem of calculating periodic solutions of (6.1) is converted into a two-point BVP defined by Equations (6.21) and (6.22).

6.4. **Continuation of Periodic Orbits.** In the previous two sections we discussed how to continue equilibria and how to compute individual periodic orbits. In this section, we will discuss how to combine those two ideas to continue the periodic orbits. The idea is to use a solution $\mathbf{x}[t, \boldsymbol{\eta}(s), \alpha(s)]$ of (6.1) in order to find the unknown period $T(s)$ such that

$$\mathbf{x}[T(s), \boldsymbol{\eta}(s), \alpha(s)] = \boldsymbol{\eta}(s). \tag{6.23}$$

Now, the system (6.1) has n equations on $n+2$ unknowns $\boldsymbol{\eta}^{\mathsf{T}}, T,$ and α, where T is the transpose. For the uniqueness of the solution to the system (6.1), we need to two more additional conditions. The first is the pseudo-arclength condition. We will discuss two possible ways to define the final condition. One of them is to use Poincaré orthogonality condition. The second involves aligning the phases of two consecutive periodic solutions.

6.4.1. *Poincare Map.* Consider the continuous time dynamical system (6.1). Let L_0 be the periodic orbit of the system (6.1). The Poincaré map is defined as the intersection of a periodic orbit in the state space of a continuous dynamical system with a certain lower-dimensional subspace (discrete dynamical system), called the Poincaré section, transversal to the flow of the system. Let $\mathbf{x}_0 \in L_0$ and $\sum$ be the smooth cross-section (See Figure 6.5) with one dimension less than the state space, $n-1$; that intersects L_0 at an non-zero angle. The transversality is where the surface of section is to be orthogonal to the flow(ϕ):

$$\sum\nolimits_{\mathbf{x}_0} := \{\mathbf{x} : \langle \phi(\mathbf{x}) | \mathbf{x} - \mathbf{x}_0 \rangle = 0\}, \text{ where } \langle \phi | g \rangle := \phi \cdot g, \text{is an inner product.}$$

Since, the Poincaré maps converts a continuous dynamical system to a discrete dynamical system. The

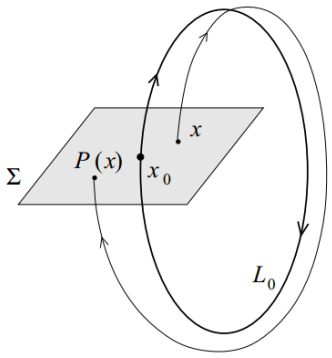

FIGURE 6.5. Poincaré section, Kuznetsov [12].

stability of a periodic orbit of the original system is closely related to the stability of the fixed point of the corresponding Poincaré map. The intersection point $\mathbf{x}_0$ is a fixed point of the Poincaré map, $\mathcal{P}$. Since, the system (6.1) is autonomous if $\mathbf{x}(t)$ is a periodic solution, then so is $\mathbf{x}(t+\omega)$, for any ω. Thus, a phase condition is needed for uniqueness. Poincaré orthogonality condition is defined as inforcing that the change in initial conditions between existing periodic orbit $\mathbf{x}_{k\ \ 1}(t)$ and the new periodic orbit $\mathbf{x}_k(t)$ orthogonal to the vector field evaluated at the initial point of $\mathbf{x}_{k\ \ 1}(t)$, that is

$$(\mathbf{x}_k(0) - \mathbf{x}_{k\ \ 1}(0))^{\mathsf{T}} \dot{\mathbf{x}}_{k\ \ 1}(0) = 0. \tag{6.24}$$

The graphical representation of Poincaré's orthogonality phase condition is shown in the Figure 6.6.

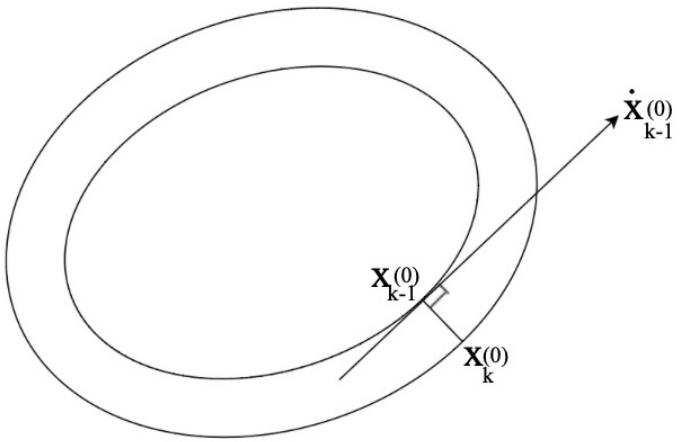

FIGURE 6.6. Poincaré orthogonality condition.

6.4.2. *Phase condition.* In practice, we do not use the condition (6.24). The most common choice is the integral phase condition, which is obtained by the solution that minimizes

$$D(\omega) := \int_0^T \|\tilde{\mathbf{x}}_k(t+\omega) - \mathbf{x}_{k\ \ 1}(t)\|^2 \mathrm{d}t, \text{ where } \tilde{\mathbf{x}}_k(t+\omega) \text{ is the solution for any } \omega. \tag{6.25}$$

Differentiating Equation (6.25) with respect to ω and setting the derivative to zero, we get

$$\int_0^T [\tilde{\mathbf{x}}_k(t+\omega) - \mathbf{x}_{k\ \ 1}(t)]^{\mathsf{T}} \dot{\tilde{\mathbf{x}}}_k(t+\omega)\mathrm{d}t = 0. \tag{6.26}$$

Let the Equation (6.26) is satisfied at $\omega = \omega\ $, that is,

$$\int_0^T [\tilde{\mathbf{x}}_k(t+\omega\ ) - \mathbf{x}_{k\ \ 1}(t)]^{\mathsf{T}} \dot{\tilde{\mathbf{x}}}_k(t+\omega\ )\mathrm{d}t = 0. \tag{6.27}$$

Let $\tilde{\mathbf{x}}_k(t+\omega\ ) \equiv \mathbf{x}_k(t)$, gives from Equation (6.27)

$$\int_0^T [\mathbf{x}_k(t) - \mathbf{x}_{k\ \ 1}(t)]^{\mathsf{T}} \dot{\mathbf{x}}_k(t)\mathrm{d}t = 0. \tag{6.28}$$

Integrating Equation (6.28) by parts, gives

$$\int_0^T \mathbf{x}_k(t)^{\mathsf{T}} \dot{\mathbf{x}}_{k\ \ 1}(t)\mathrm{d}t = \int_0^T \mathbf{x}_k(t)^{\mathsf{T}} \mathbf{F}(\mathbf{x}_{k\ \ 1}(t))\mathrm{d}t = 0. \tag{6.29}$$

Condition (6.29) is called the integral phase condition, which is the generalization of Poincaré phase condition for all consecutive solutions of the system. The second condition is given by the pseudo-arclength constraint (continuation equation)

$$\int_0^T [\mathbf{x}_k(t) - \mathbf{x}_{k\ \ 1}(t)]^{\mathsf{T}} \dot{\mathbf{x}}_{k\ \ 1}(t)\mathrm{d}t + (T_k - T_{k\ \ 1})\dot{T}_{k\ \ 1} + (\alpha_k - \alpha_{k\ \ 1})\dot{\alpha}_{k\ \ 1} - \Delta s = 0. \tag{6.30}$$

6.5. **Modification for Hamiltonian System.** In this section, we will discuss about the theory of continuation in Hamiltonian systems. More specifically, modification of the system (2.4) to Hamiltonian system. Hamiltonian systems are quite different from the dissipative systems. In dissipative system periodic orbits are generally isolated, and therefore an external parameter is required to continue periodic orbits. The idea of continuation in dissipative system is convert the initial value problem into a boundary value problem by making use of two parameters, period and the external parameter. In Hamiltonian systems conserved quantities and symmetries are related by Noether's theorem, and they result in families of non-isolated periodic orbits. The absence of the internal parameter in the Hamiltonian systems causes general continuation scheme to fail. Therefore, a modification of the system (2.4) is essential in the analysis of the continuation of periodic orbits in a Hamiltonian system. Before making any modification to the system (2.4), we will define a few notations and definitions.

(6.31) Let $\mathcal{F} := \{F \in \mathcal{C}^1(\mathsf{R}^{2n}, \mathsf{R}) : \{F, \mathcal{H}\} \equiv 0\}$ denote the space of all first integrals of system (2.4).

Let $p_0 \in \mathsf{R}^{2n}$ generates a nontrivial periodic orbit $\Gamma_0 := \{\phi_H^t(p_0) \mid t \in \mathsf{R}\}$ of the system (2.4), with a minimal period $T_0 > 0$.

(6.32) $$\text{Let } W := \{\nabla F(p_0) : F \in \mathcal{F}\}, \text{ and } \mathcal{G}_0(T, p) := \phi_H(T, p) - p = 0.$$

Definition 5 (Normal periodic orbit [4]). *The periodic orbit Γ_0 generated by p_o is normal if*

(6.33) $$\operatorname{Im} D\mathcal{G}_0(T_0, p_0) = \operatorname{Im}(\mathcal{M} - \mathcal{I}) + \operatorname{Re} X_H(p_0) = W^?,$$

where $\mathcal{M}$ monodromy matrix of the periodic orbit Γ_0.

Definition 6 (Submersitivity [15]). *If $G_0 : \mathsf{R}^m \to \mathsf{R}^n$ with $m > n$ is a smooth mapping, and if $x_0 \in \mathsf{R}^m$ is a solution of*

(6.34) $$G_0(x) = 0$$

such that $DG_0(x_0) \in \mathcal{L}(\mathsf{R}^m; \mathsf{R}^n)$ is surjective (i.e. G_0 is a submersion at x_0,) then near x_0 the solution set (6.34) is a smooth $(m-n)$-dimensional submanifold of R^m.

If Γ_0 is a normal periodic orbit, then by definition of normality, $\mathcal{G}_0$ is not subversive at the point (T_0, p_0). To take care of the lack of submersivity (bijectivity), we add extra terms to the system (2.4). The new modified system is

(6.35) $$\dot{u} = X_H(u) + \sum_{i=1}^{k} \alpha_i \nabla F_i(u)$$

where $F_i \in \mathcal{F}$ are chosen in such a way that $\{\nabla F_i(p_0), 1 \le i \le k\}$ forms a basis for W. The idea of the continuation of a periodic solution of (6.35) is to look for the solutions of the equation

(6.36) $$G(T, p, \alpha) := \phi_H(T, p, \alpha) - p = 0.$$

Let $u(t)$ is a T-periodic solution of the system (6.35), and let $F(u) := \sum_{i=1}^{k} \alpha_i F_i(u)$. Then, we have,

(6.37) $$\frac{\mathrm{d}}{\mathrm{d}t} F(u(t)) = \|\nabla F(u(t))\|^2.$$

Integrating both of Equation (6.37) with respect to t, we get

(6.38) $$\int_0^T \|\nabla F(u(t))\|^2 \mathrm{d}t = F(u(T)) - F(u(0)) = 0 \implies \nabla F(u(t)) = 0, \forall t \in \mathsf{R}.$$

From the Equation (6.38), $\nabla F_j(p_0)$ are linearly independent $\forall j$. Therefore, if $u(0)$ is taken sufficiently close to p_0, then the same argument shows that $\nabla F_j(u(0))$ are linearly independent $\forall j$. The linear independency of $\nabla F_j(p_0)$ shows that the system (6.35) can only have a periodic orbit near Γ_0 if $\alpha = 0$. That means, if $\alpha = 0$, then the periodic orbits of the system (6.35) are the same as the periodic orbits of the Hamiltonian system (2.4).

7. Continuation Software

In this thesis proposal, we have used MATLAB for the continuation of the periodic orbits. We wrote a MATLAB code for the periodic orbit continuation of the Lorentz system and the Rössler system, in which we made use of the shooting method discussed in section 6.3. The computer program AUTO [3] is widely regarded as the best tool for the computation of periodic orbits, and we plan to use it for all our future computations.

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